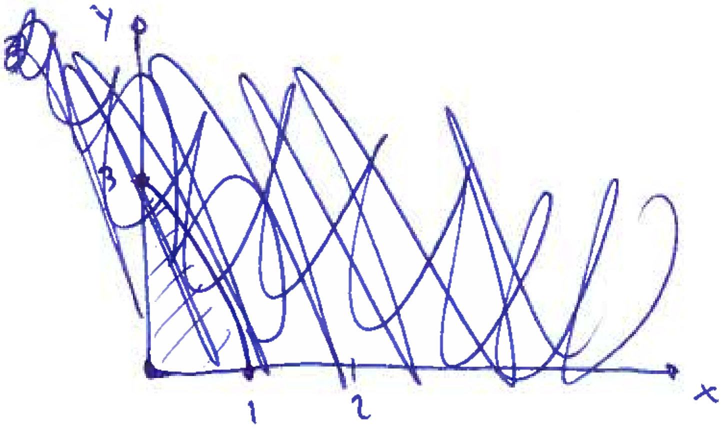
Problem 3-07

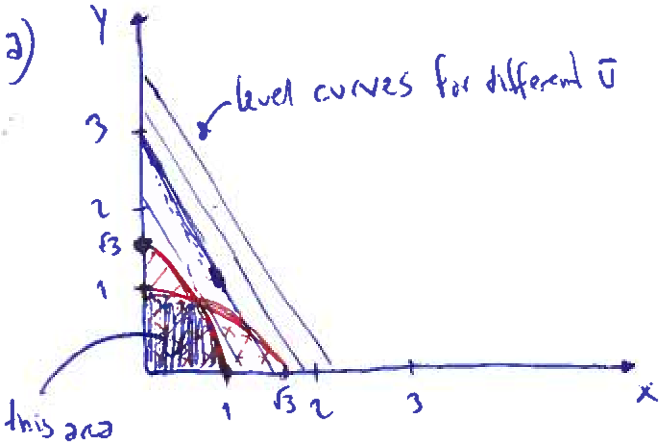
Seminar 4 - ECON 4140

$$\max (2x+y) \text{ s.t. } \begin{cases} (x+1)^2 + y^2 \le 4 \\ x^2 + (y+1)^2 \le 4 \end{cases} \quad x \ge 0,\ y \ge 0$$



$f(x,y) : 2x + y = \bar{U} \Rightarrow \underline{y = \bar{U} - 2x}$

a)



Draw the constraint curves by checking extreme cases $x=0$, and $y=0$, then draw smooth curve between (the constraints are of a so called elliptic form, that is the equations are elliptic).

b) Want x and y as large as possible combined

$\Rightarrow$ Both constraints are binding. $\bar{U}$ is higher the further out in the plane one is

$\Rightarrow$ the point in the figure where the two constraint curves crosses is optimal.

Since the two constraints are symmetric, this is when $x = y$, and thus when

$(x+1)^2 + x^2 = 4 \quad 2x^2 + 2x + 1 = 4 \quad x^2 + x - \frac{3}{2} = 0 \quad \Rightarrow \underline{x = y = \frac{1}{2}(\sqrt{7} - 1)}$

c) $\mathcal{L} = 2x + y - \lambda_1((x+1)^2 + y^2 - 4) - \lambda_2(x^2 + (y+1)^2 - 4) + \mu_1 x + \mu_2 y$

$$\boxed{\begin{array}{ll}
(1) & \frac{\partial \mathcal{L}}{\partial x} = 2 - 2\lambda_1(x+1) - 2\lambda_2 x + \mu_1 = 0 \\
(2) & \frac{\partial \mathcal{L}}{\partial y} = 1 - 2\lambda_1 y - 2\lambda_2(y+1) + \mu_2 = 0 \\
(3) & \lambda_1 \geq 0 \quad (\lambda_1 = 0 \text{ if } (x+1)^2 + y^2 < 4) \\
(4) & \lambda_2 \geq 0 \quad (\lambda_2 = 0 \text{ if } x^2 + (y+1)^2 < 4) \\
(5) & \mu_1 \geq 0 \quad (\mu_1 = 0 \text{ if } x > 0) \\
(6) & \mu_2 \geq 0 \quad (\mu_2 = 0 \text{ if } y > 0) \\
(7) & (x+1)^2 + y^2 \leq 4 \\
(8) & x^2 + (y+1)^2 \leq 4 \\
(9) & x \geq 0 \\
(10) & y \geq 0
\end{array}}$$

(7)-(10) satisfied. And we note by (5) and (6) that $\underline{\mu_1 = 0}$ and $\underline{\mu_2 = 0}$.

$\Rightarrow$ (5) and (6) satisfied. Need to check (1)-(4).

(1) gives $\underline{\lambda_2 = \frac{1 - \lambda_1(x+1)}{x}}$ which in (2) gives $1 - 2\lambda_1 y = 2(y+1)\frac{1-\lambda_1(x+1)}{x}$

$\Rightarrow \quad x - 2\lambda_1 xy = 2(y+1) - 2(y+1)(x+1)\lambda_1$

$\Rightarrow \quad 2\lambda_1[(x+1)(y+1) - xy] = 2(y+1) - x \quad \Rightarrow \quad \underline{\lambda_1 = \frac{y+1-\frac{1}{2}x}{x+y+1}}$

With $x = y = \frac{1}{2}(\sqrt{7}-1)$ this gives $\lambda_1 = \frac{\frac{\sqrt{7}}{2} - \frac{1}{2} - \frac{\sqrt{7}}{4} + \frac{1}{4}}{\sqrt{7}} = \frac{\frac{\sqrt{7}}{2} - \frac{1}{2}}{2\sqrt{7}} = \frac{\sqrt{7}-1}{4\sqrt{7}}$

So $\underline{\lambda_1 = \frac{\sqrt{7}-1}{4\sqrt{7}} > 0}$ and $\lambda_2 = \frac{1 - \frac{\sqrt{7}-1}{4\sqrt{7}}(\frac{1}{2}(\sqrt{7}-1)+1)}{\frac{1}{2}(\sqrt{7}-1)} = \frac{4\sqrt{7} - (\sqrt{7}-1)[\frac{1}{2}(\sqrt{7}-1)+1]}{2\sqrt{7}(\sqrt{7}-1)}$

$\underline{\lambda_2 = \frac{4\sqrt{7}-3}{2\sqrt{7}(\sqrt{7}-1)} > 0}$ $\quad = \frac{4\sqrt{7} - \frac{1}{2}(\sqrt{7}-1)^2 - \sqrt{7} + 1}{2\sqrt{7}(\sqrt{7}-1)} = \frac{3\sqrt{7} + 1 - \frac{7}{2} + \sqrt{7} - \frac{1}{2}}{2\sqrt{7}(\sqrt{7}-1)} = \underline{\frac{4\sqrt{7}-3}{2\sqrt{7}(\sqrt{7}-1)}}$

(1)-(4) is also satisfied. $\quad \Rightarrow$ <u>The Kuhn-Tucker conditions are satisfied</u>

d) The constraint corresponding to λ_2 is changed.

By the envelope theorem, the approximate change is

$$\lambda_2 \cdot 0{,}1 = \frac{4\sqrt{7}-3}{2\sqrt{7}(\sqrt{7}-1)} \cdot 0{,}1$$

Problem 3-10

$$f(x,y) = -x^4 - cx^2 + 6xy - 6y^2$$

a) $f'_x = -4x^3 - 2cx + 6y \qquad f'_y = 6x - 12y$

$f''_{xx} = -12x^2 - 2c \leq 0 \quad \underline{c \geq -6x^2} \qquad f''_{yy} = -12 < 0 \quad f''_{xy} = 6$

$$f''_{xx} f''_{yy} - (f''_{xy})^2 = -2(6x^2+c)(-12) - 36 = 24(6x^2+c) - 36 \geq 0$$

$6x^2 + c \geq \frac{3}{2} \qquad \underline{c \geq \frac{3}{2} - 6x^2}$ Last condition most strict, and strictest possible is when $x=0$, then $c \geq \frac{3}{2}$

$\Rightarrow$ When $c \geq \frac{3}{2}$, f is concave in the whole plane

b) $\mathcal{L}(x,y) = -x^4 - y^4 - 4x^2 + 6xy - 6y^2 + ax + by - \lambda(x+y^2-1) - \mu(-1-y)$

(1) $\frac{\partial \mathcal{L}}{\partial x} = -4x^3 - 8x + 6y + a - \lambda = 0$

(2) $\frac{\partial \mathcal{L}}{\partial y} = -4y^3 + 6x - 12y + b - 2\lambda y + \mu = 0$

(3) $\lambda \geq 0 \quad (\lambda = 0 \text{ if } x + y^2 = 1)$

(4) $\mu \geq 0 \quad (\mu = 0 \text{ if } y = -1)$

(5) $x + y^2 \leq 1$

(6) $y \geq -1$

c) Want (5) and (6) to be binding.

Then (1) gives $\underline{-6+a=\lambda}$ $\Rightarrow$ need $\boxed{a \geq 6}$

(2) gives $4+12+b+2(-6+a)+\mu=0$

$\mu=-b-4-2a\geq 0$ $\Rightarrow$ $\boxed{2a+b\leq -4}$

Since $f(x,y)$ from a) is concave

when $c\geq\frac{3}{2}$, $\mathcal{L}(x,y)$ is concave since we have $-4x^2$ when $4>\frac{3}{2}$.

$\Rightarrow$ $\underline{\underline{(x,y)=(0,-1) \text{ is indeed optimal}}}$

Problem 3-13

max $a\ln(z+1)-z-2x-y$ s.t. $z^2\leq x+y$ and $x\geq 0, y\geq 0, z\geq 0$.

a) $\mathcal{L}(x,y,z)=a\ln(z+1)-z-2x-y-\lambda(z^2-x-y)$

(1) $\frac{\partial\mathcal{L}}{\partial x}=-2+\lambda\leq 0$ $(=0 \text{ if } x>0)$

(2) $\frac{\partial\mathcal{L}}{\partial y}=-1+\lambda\leq 0$ $(=0 \text{ if } y>0)$

(3) $\frac{\partial\mathcal{L}}{\partial z}=\frac{a}{z+1}-1-2\lambda z\leq 0$ $(=0 \text{ if } z>0)$

(4) $\lambda\geq 0$ $(\lambda=0 \text{ if } z^2<x+y)$

(5) $z^2\leq x+y$

(6) $x\geq 0$

(7) $y\geq 0$

(8) $z\geq 0$

b) Notice: cost of high x bigger than high y, and except for that they appear symmetric $\Rightarrow$ deduce $\underline{x=0}$, $y \geq 0$.
Are then left with three constraints to check.

All binding
- $z^2 = y$, $y=0$, $z=0$
  (2) gives $\lambda \leq 1$, (3) gives $a \leq 1$ $\Rightarrow$ $\boxed{\text{ok if } a \leq 1}$

two binding
- $z^2 = y$, $y=0$, $z>0$ $\boxed{\text{Impossible}}$
- $z^2 = y$, $y>0$, $z=0$ $\boxed{\text{Impossible}}$
- $z^2 < y$, $y=0$, $z=0$ $\boxed{\text{Impossible}}$

one binding
- $\underline{z^2 = y}$ $y>0$ $z>0$

(2) gives $\underline{\lambda = 1}$ (1) gives $-2 \leq -1$ $\underline{\text{ok}}$

(3) gives $\frac{a}{z+1} - 1 - 2z = 0$ $\quad a - z - 1 - 2z^2 - 2z = 0$ $\quad 2z^2 + 3z + 1 - a = 0$

$z^2 + \frac{3}{2}z + \frac{1}{2} - \frac{a}{2} = 0$ $\quad z = \frac{-\frac{3}{2} \pm \sqrt{\frac{9}{4} - (2-2a)}}{2} = \frac{-\frac{3}{2} \pm \sqrt{2a + \frac{1}{4}}}{2}$

Not possible when $a \leq 1$, when $a > 1$ then $\underline{z = -\frac{3}{4} + \frac{\sqrt{2a+\frac{1}{4}}}{2}}$

Then $\underline{y = \frac{2a+\frac{1}{4}}{2} - \frac{3+\sqrt{2a+\frac{1}{4}}}{4} + \frac{9}{16}} > 0$ $\quad a - \frac{\sqrt{2a+\frac{1}{4}}}{4} > \frac{1}{16}$

$\Rightarrow a > \frac{1}{16} + \frac{\sqrt{2a+\frac{1}{4}}}{4} > \frac{1}{16} + \frac{\sqrt{\frac{9}{4}}}{4} = \frac{1}{16} + \frac{3}{8} = \frac{7}{16}$ ok.

- $z^2 < y$ $y=0$ $z>0$ $\boxed{\text{Impossible}}$
- $z^2 < y$ $y>0$ $\underline{z=0}$ $\Rightarrow$ $\underline{\lambda = 0}$ (2) gives $-1 = 0$ $\boxed{\text{Impossible}}$

none binding
- $z^2 < y$ $y>0$ $z>0$ $\Rightarrow$ $\underline{\lambda = 0}$ (2) gives $-1 = 0$ $\boxed{\text{Impossible}}$

$\Rightarrow$ The solution is $(x,y,z) = (0,0,0)$ when $a \leq 1$ and
$(x,y,z) = \left(0, \frac{2a+\frac{1}{4}}{2} - \frac{3+\sqrt{2a+\frac{1}{4}}}{4} + \frac{9}{16}, -\frac{3}{4} + \frac{\sqrt{2a+\frac{1}{4}}}{2}\right)$ when $a > 1$

c) $\mathcal{L}''_{xx} = 0$ $\quad \mathcal{L}''_{xy} = 0$ $\quad \mathcal{L}''_{xz} = 0$
$\mathcal{L}''_{yx} = 0$ $\quad \mathcal{L}''_{yy} = 0$ $\quad \mathcal{L}''_{yz} = 0$ $\quad \Rightarrow$ $\underline{\mathcal{L} \text{ concave}}$
$\mathcal{L}''_{zx} = 0$ $\quad \mathcal{L}''_{zy} = 0$ $\quad \mathcal{L}''_{zz} = -\frac{a}{(z+1)^2} - 2\lambda < 0$

$\Rightarrow$ Solution is indeed the optimal solution

Exam 2011, Problem 3

a) $f(x_1,\dots,x_n) = (2\pi)^{-\frac{n}{2}} e^{\frac{-(x_1^2+\dots+x_n^2)}{2}}$

$e^{\cdot}$ increasing, and $-(x_1^2+\dots+x_n^2)$ concave $\Rightarrow$ is also quasiconcave

$\Rightarrow$ $\underline{\underline{f(x_1,\dots,x_n) \text{ quasiconcave}}}$

b) $\mathcal{L}(x,y) = \frac{1}{2\pi} e^{\frac{-(x^2+y^2)}{2}} - \lambda(-y) - \mu(y-(x-a)^{2011})$

$$\boxed{\begin{array}{ll}
(1) & \frac{\partial \mathcal{L}}{\partial x} = -\frac{x}{2\pi} e^{\frac{-(x^2+y^2)}{2}} + 2011\mu(x-a)^{2010} = 0 \\
(2) & \frac{\partial \mathcal{L}}{\partial y} = -\frac{y}{2\pi} e^{\frac{-(x^2+y^2)}{2}} + \lambda - \mu = 0 \\
(3) & \lambda \geq 0 \quad (\lambda = 0 \text{ if } y > 0) \\
(4) & \mu \geq 0 \quad (\mu = 0 \text{ if } y < (x-a)^{2011}) \\
(5) & y \geq 0 \\
(6) & y \leq (x-a)^{2011}
\end{array}}$$

- Both binding $\underline{y=0}$ $y=(x-a)^{2011}$ $\Rightarrow$ $\underline{x=a}$
  (2) gives $\underline{\lambda=\mu}$ (1) gives $-\frac{a}{2\pi}e^{-\frac{a^2}{2}} = 0$ Impossible
- $\underline{y=0}$ $y<(x-a)^{2011}$ $\Rightarrow$ $\underline{x>a}$ $\Rightarrow$ $\underline{\mu=0}$
  (1) gives $x=0$ Impossible (Since $a>0$)
- $y>0$ $\underline{y=(x-a)^{2011}}$ $\Rightarrow$ $x>0$, $\underline{\lambda=0}$ (2) gives $\mu = -\frac{y}{2\pi}e^{\frac{-(x^2+y^2)}{2}} < 0$ Impossible
- None binding $y>0$ $y<(x-a)^{2011}$ $\Rightarrow$ $\underline{\lambda=0}$ $\underline{\mu=0}$ $\Rightarrow$ (1) gives $\underline{x=0}$
  (2) gives $y=0$ Impossible

<u>No candidates</u>

c) Optimizing f by choosing x and y as small as possible, that is choose both binding.
$\Rightarrow$ $(x^*,y^*)=(a,0)$. $\nabla g_1 = (0,-1)$ $\nabla g_2 = (-2011(x-a)^{2010}, 1)$

gradient of first constraint

gradient of second constraint

with $(x^*,y^*)=(a,0)$ we get $\nabla g_1 = (0,-1)$ $\nabla g_2 = (0,1) = -\nabla g_1$
Since $\nabla g_2 = -\nabla g_1$, they are not linearly independent $\Rightarrow$ <u>Constraint Qualification violated!</u>

$\boxed{\text{Problem on webpage}}$

$V(q) = \max -\|x\| n^{\frac{1}{2}}$ s.t. $\frac{w(x_1)+\dots+w(x_n)}{n} \le q$

where $w(x_i) = |x_i - 1| + 3|x_i - 2| + e^{x_i - 1} - 4$

Solution is of the form $x^* = k(1,1,\dots,1)$ where $k>0$. (due to symmetry of problem).

a)

$\mathcal{L}(x) = -\|x\| n^{\frac{1}{2}} - \lambda\left(\frac{w(x_1)+\dots+w(x_n)}{n} - q\right)$

$$\boxed{\begin{array}{ll} (1) & \frac{\partial \mathcal{L}}{\partial x_i} = -\|x\|^{-1} x_i n^{\frac{1}{2}} - \lambda \frac{w'(x_i)}{n} = 0 \\ (2) & \lambda \ge 0 \quad (\lambda = 0 \text{ if } \frac{w(x_1)+\dots+w(x_n)}{n} < q) \\ (3) & \frac{w(x_1)+\dots+w(x_n)}{n} \le q \end{array}}$$

$i = 1,\dots,n$

(1) gives $\lambda = -\frac{x_i n^{\frac{3}{2}}}{\|x\| w'(x_i)} = -\frac{k n^{\frac{3}{2}}}{k w'(k)} = -\frac{n^{\frac{3}{2}}}{w'(k)}$

$= \frac{-n^{\frac{3}{2}}}{-1-3+e^{k-1}} = \frac{n^{\frac{3}{2}}}{4-e^{k-1}} \rightarrow \frac{n^{\frac{3}{2}}}{4-\frac{1}{e}}$ (Since $V'(q) = \lambda$)

b) $\lambda = \frac{-n^{\frac{3}{2}}}{w'(k)} = \frac{-n^{\frac{3}{2}}}{1-3+e^{k-1}} \rightarrow \frac{-n^{\frac{3}{2}}}{-2+1} = n^{\frac{3}{2}}$ (Remember want k small as possible)

c) w not differentiable in 0, but know $\lambda = V'(0)$.

$\Rightarrow \lambda^+ = \frac{n^{\frac{3}{2}}}{4-\frac{1}{e}}$ $\quad \lambda^- = n^{\frac{3}{2}}$ $\quad \Rightarrow \lambda \in \left[\frac{n^{\frac{3}{2}}}{4-\frac{1}{e}}, n^{\frac{3}{2}}\right]$ ok

d) (1) gives $-n^{\frac{1}{2}} = \lambda \frac{w'(1)}{n}$ $\quad \lambda = \frac{-n^{\frac{3}{2}}}{w'(1)} = \begin{cases} n^{\frac{3}{2}} & \text{This one ok.} \\ \frac{1}{3} n^{\frac{3}{2}} & \text{This one also in interval} \end{cases}$

(2) ok.

(3) gives $\frac{n w(1)}{n} = w(1) \le 0$ That is $3-3=0 \le 0$ ok.

$\mathcal{L}(x)$ concave $\Rightarrow$ Is indeed solution

$\boxed{\text{Problem 1-06}}$

a) $$|D_t| = t\begin{vmatrix} 2 & t & 3 \\ -2 & t & 0 \\ 1 & 0 & 3 \end{vmatrix} - \begin{vmatrix} 0 & 2 & t \\ 1 & -2 & t \\ 2t & 1 & 0 \end{vmatrix} = t\left[2\begin{vmatrix} t & 0 \\ 0 & 3 \end{vmatrix} - t\begin{vmatrix} -2 & 0 \\ 1 & 3 \end{vmatrix} + 3\begin{vmatrix} -2 & t \\ 1 & 0 \end{vmatrix}\right] - \left[-2\begin{vmatrix} 1 & t \\ 2t & 0 \end{vmatrix} + t\begin{vmatrix} 1 & -2 \\ 2t & 1 \end{vmatrix}\right]$$

$$= t\left[6t + 6t - 3t\right] - \left[4t^2 + t(1+4t)\right] = 9t^2 - 8t^2 - t = t^2 - t = \underline{t(t-1)}$$

$\Rightarrow$ $\boxed{\text{Rank} = 4 \text{ when } t \neq 0, t \neq 1}$

$$t = 0 \Rightarrow \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 3 \\ 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \overset{+2-2}{\sim} \begin{pmatrix} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

↳ Rank 2

$\Rightarrow$ $\underline{\underline{\text{Rank} = 2 \text{ when } t = 0}}$

$\underline{\underline{t = 1 \Rightarrow \text{Rank} = 3}}$

b) $CB + CXA^{-1} = A^{-1}$ $\quad CXA^{-1} = A^{-1} - CB$ $\quad CX = I - CBA$ $\quad \underline{\underline{X = C^{-1} - BA}}$

$\boxed{\text{Problem 1-07}}$

a) $AA^{-1} = I \Rightarrow$ $$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} = \begin{pmatrix} ax_{11} & ax_{12} & ax_{13} \\ bx_{21} & bx_{22} & bx_{23} \\ cx_{31} & cx_{32} & cx_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\Rightarrow x_{11} = \frac{1}{a}$ $\quad x_{22} = \frac{1}{b}$ $\quad x_{33} = \frac{1}{c}$ $\quad$ Rest $= 0$

$$\Rightarrow \underline{\underline{A^{-1} = \begin{pmatrix} \frac{1}{a} & 0 & 0 \\ 0 & \frac{1}{b} & 0 \\ 0 & 0 & \frac{1}{c} \end{pmatrix}}}$$

b) DONE IN SEMINAR 2!

$\boxed{\text{Problem 1-13}}$

ALSO DONE IN SEMINAR 2!