$\boxed{1\text{-}03}$

a) We have $(a-c)=(a-b)+(b-c)$, and thus they are not linearly independent.

b) $d = 4a - b - c$

$$x(a-b)+y(b-c)+z(a-c) = (x+z)a + (-x+y)b + (-y+z)c$$
$$= 4a - b - c$$

$\Rightarrow$ $\underline{x+z=4}$ $\quad$ $\underline{-x+y=-1}$ $\quad$ $-(y+z)=-1$ $\quad$ $\underline{y+z=1}$

$x = 4-z$ $\quad$ $-4+z+y=-1$ $\quad$ $y=3-z$ $\quad$ $3-z+z=1$ $\quad$ $3=1$ Impossible

it is not possible

$\boxed{1\text{-}04}$

a) $\begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 4 = 0$

$(1-\lambda)^2=4$ $\quad$ $1-\lambda=\pm 2$ $\quad$ $\lambda = \begin{cases} 3 \\ -1 \end{cases}$ $\quad$ $\lambda_1=3$ $\quad$ $\lambda_2=-1$

$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 3\begin{pmatrix} x \\ y \end{pmatrix}$ $\quad$ $\begin{pmatrix} x+2y \\ 2x+y \end{pmatrix} = \begin{pmatrix} 3x \\ 3y \end{pmatrix}$ $\quad$ $2x=2y$ $\quad$ $\underline{x=y}$

$3y=3y$

$\Rightarrow$ $x_1 = t\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, t any number

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -\begin{pmatrix} x \\ y \end{pmatrix} \qquad \begin{pmatrix} x+2y \\ 2x+y \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix} \qquad \begin{matrix} 2x+2y=0 & \underline{x=-y} \\ \underline{-y=-y} & \end{matrix}$$

$$\Rightarrow \quad \underline{\underline{x_2 = t\begin{pmatrix} -1 \\ 1 \end{pmatrix}, \ t \text{ any number}}}$$

b) $x_0 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1{,}5 \\ 1{,}5 \end{pmatrix} + \begin{pmatrix} -0{,}5 \\ 0{,}5 \end{pmatrix}$ (these are eigenvectors of A)

$$x_{t+1} = Ax_t \qquad x_1 = Ax_0 = A\begin{pmatrix} 1{,}5 \\ 1{,}5 \end{pmatrix} + A\begin{pmatrix} -0{,}5 \\ 0{,}5 \end{pmatrix} = \lambda_1\begin{pmatrix} 1{,}5 \\ 1{,}5 \end{pmatrix} + \lambda_2\begin{pmatrix} -0{,}5 \\ 0{,}5 \end{pmatrix}$$

$$x_2 = Ax_1 = \lambda_1 A\begin{pmatrix} 1{,}5 \\ 1{,}5 \end{pmatrix} + \lambda_2 A\begin{pmatrix} -0{,}5 \\ 0{,}5 \end{pmatrix} = \lambda_1^2\begin{pmatrix} 1{,}5 \\ 1{,}5 \end{pmatrix} + \lambda_2^2\begin{pmatrix} -0{,}5 \\ 0{,}5 \end{pmatrix}$$

$$\Rightarrow \quad x_t = \lambda_1^t\begin{pmatrix} 1{,}5 \\ 1{,}5 \end{pmatrix} + \lambda_2^t\begin{pmatrix} -0{,}5 \\ 0{,}5 \end{pmatrix} = \underline{\underline{3^t\begin{pmatrix} 1{,}5 \\ 1{,}5 \end{pmatrix} + (-1)^t\begin{pmatrix} -0{,}5 \\ 0{,}5 \end{pmatrix}}}$$

$\boxed{1\text{-}06}$

$$|D_t| = \begin{vmatrix} t & 0 & 0 & 1 \\ 0 & 2 & t & 3 \\ 1 & -2 & t & 0 \\ 2t & 1 & 0 & 3 \end{vmatrix} = t\begin{vmatrix} 2 & t & 3 \\ -2 & t & 0 \\ 1 & 0 & 3 \end{vmatrix} - \begin{vmatrix} 0 & 2 & t \\ 1 & -2 & t \\ 2t & 1 & 0 \end{vmatrix} = t\left[6t - t\begin{vmatrix} -2 & 0 \\ 1 & 3 \end{vmatrix} - 3t\right] - \left[-2\begin{vmatrix} 1 & t \\ 2t & 0 \end{vmatrix} + t\begin{vmatrix} 1 & -2 \\ 2t & 1 \end{vmatrix}\right]$$

$$= 9t^2 - 4t^2 - t - 4t^2 = \underline{t(t-1)}$$

Thus, for $t \neq 0$ and $t \neq 1$, the rank is 4

For $t=0$, we have $D_0 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 3 \\ 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 3 \end{pmatrix}$, where for instance

$$\begin{vmatrix} 0 & 2 & 3 \\ 1 & -2 & 0 \\ 0 & 1 & 3 \end{vmatrix} = -2\begin{vmatrix} 1 & 0 \\ 0 & 3 \end{vmatrix} + 3\begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} = -6 + 3 \neq 0, \text{ thus,}$$

for $t=0$ the rank is 3

For $t=1$, $D_t = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 1 & 3 \\ 1 & -2 & 1 & 0 \\ 2 & 1 & 0 & 3 \end{pmatrix}$

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 1 & -2 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ -2 & 1 \end{vmatrix} = 2+2 = 4 \neq 0$$

Thus, for $t=1$ the rank is 3

b) $CB + CXA^{-1} = A^{-1}$ $\qquad B + XA^{-1} = C^{-1}A^{-1}$

$XA^{-1} = C^{-1}A^{-1} - B$ $\qquad X = C^{-1} - BA$

$\boxed{1\text{-}12}$

$$|D(s)| = \begin{vmatrix} 1 & 2s & 1 & 1 \\ -2 & 1 & -2 & 3s \\ 1 & 1-s & -1 & 5 \\ -1 & 2 & s & -3 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 3s \\ 1-s & -1 & 5 \\ 2 & s & -3 \end{vmatrix} - 2s\begin{vmatrix} -2 & -2 & 3s \\ 1 & -1 & 5 \\ -1 & s & -3 \end{vmatrix} + \begin{vmatrix} -2 & 1 & 3s \\ 1 & 1-s & 5 \\ -1 & 2 & -3 \end{vmatrix} - \begin{vmatrix} -2 & 1 & -2 \\ 1 & 1-s & -1 \\ -1 & 2 & s \end{vmatrix}$$

$$= \begin{vmatrix} -1 & 5 \\ s & -3 \end{vmatrix} + 2\begin{vmatrix} 1-s & 5 \\ 2 & -3 \end{vmatrix} + 3s\begin{vmatrix} 1-s & -1 \\ 2 & s \end{vmatrix} - 2s\left[-2\begin{vmatrix} -1 & 5 \\ s & -3 \end{vmatrix} + 2\begin{vmatrix} 1 & 5 \\ -1 & -3 \end{vmatrix} + 3s\begin{vmatrix} 1 & -1 \\ -1 & s \end{vmatrix}\right]$$

$$-2\begin{vmatrix} 1-s & 5 \\ 2 & -3 \end{vmatrix} - \begin{vmatrix} 1 & 5 \\ -1 & -3 \end{vmatrix} + 3s\begin{vmatrix} 1 & 1-s \\ -1 & 2 \end{vmatrix} - \left[-2\begin{vmatrix} 1-s & -1 \\ 2 & s \end{vmatrix} - \begin{vmatrix} 1 & -1 \\ -1 & s \end{vmatrix} - 2\begin{vmatrix} 1 & 1-s \\ -1 & 2 \end{vmatrix}\right]$$

$$= 3-5s + 2[3s-3-10] + 3s(s-s^2+2) - 2s[-6+10s \overset{+4}{\cancel{}} + 3s^2 - 3s]$$

$$+6 - 6s + 20 + 3 - 5 + 6s + 3s - 3s^2 - [-2s + 2s^2 - 4 - s + 1 - 4 - 2 + 2s]$$

$$= s^3[-3-6] + s^2[3-20+6-3-2] + s[-5+6+6+4-6+6+3+1]$$

$$+3 - 26 + 24 + 9 = -9s^3 - 16s^2 + 15s + 10$$

Thus, $D(s)$ has rank 4 as long as $9s^3 + 16s^2 - 15s - 10 \neq 0$.

~~For $s=1$, $|D(1)| = 0$~~

· For $s = 1$, $|D(1)| = 0$.

However $\begin{vmatrix} 1 & 2 & 1 \\ -2 & 1 & -2 \\ 1 & 0 & -1 \end{vmatrix} = \begin{vmatrix} 1 & -2 \\ 0 & -1 \end{vmatrix} - 2\begin{vmatrix} -2 & -2 \\ 1 & -1 \end{vmatrix} + \begin{vmatrix} -2 & 1 \\ 1 & 0 \end{vmatrix} = -1 - 8 - 1 = -10 \neq 0$

thus, $\underline{\underline{D(1) \text{ has rank } 3}}$

b) This is the equationsystem corresponding to $D(1)$.

Since $D(1)$ has rank 3, the equationsystem has 1 degree of freedom.