Seminar 5 - ECON 4140

Problem 1, Exam 11

$$A = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 4 \end{pmatrix}$$

a) $Av_1 = \lambda v_1$

$$Av_1 = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \\ 10 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \Rightarrow \underline{\underline{\lambda = 5}}$$

b) $$\begin{pmatrix} 4 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4x \\ 4y \\ 4z \end{pmatrix} \quad \begin{pmatrix} 4x+y \\ x+2z \\ 2y+4z \end{pmatrix} = \begin{pmatrix} 4x \\ 4y \\ 4z \end{pmatrix} \quad \Rightarrow y=0, \quad x=-2z$$

$$\Rightarrow \underline{\underline{v_2 = t\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}}} \qquad t \text{ any number.}$$

c) Trace of matrix equals sum of eigenvalues

$$\Rightarrow \lambda_1 + \lambda_2 + \lambda_3 = 5 + 4 + \lambda_3 = 4 + 4 = 8 \quad \Rightarrow \underline{\underline{\lambda_3 = -1}}$$

d) A is indefinite as the eigenvalues have both positive and negative signs.

Problem 1-05

$$|A_a| = \lambda_1\lambda_2\lambda_3 = 2a^2 \qquad \text{Trace } A = \lambda_1 + \lambda_2 + \lambda_3 = 2(a+1)$$

$Av_1 = \lambda v_1$ $$\begin{pmatrix} 4 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \lambda x \\ \lambda y \\ \lambda z \end{pmatrix} \quad \begin{pmatrix} 4x+y \\ x+2z \\ 2y+4z \end{pmatrix} = \begin{pmatrix} \lambda x \\ \lambda y \\ \lambda z \end{pmatrix}$$

$\underline{y = x(\lambda - 4)}$ $\quad x + 2z = \lambda x(\lambda - 4)$ $\quad \underline{z = \frac{1}{2}x[\lambda(\lambda-4)-1]}$

$2x(\lambda-4) + 2x[\lambda(\lambda-4)-1] = \frac{\lambda}{2}[\lambda(\lambda-4)-1]$

$$\begin{vmatrix} 2a-\lambda & 0 & 0 \\ 0 & -\lambda & -a \\ 2-a & 1 & 2-\lambda \end{vmatrix} = (2a-\lambda)\begin{vmatrix} -\lambda & -a \\ 1 & 2-\lambda \end{vmatrix} = (2a-\lambda)\left[\lambda(\lambda-2)+a\right]$$

$$= 2a\lambda(\lambda-2) + 2a^2 - \lambda^2(\lambda-2) - \lambda a = 0$$

$$\lambda(\lambda-2)(2a-\lambda) + a(2a-\lambda) = 0 \qquad (2a-\lambda)(\lambda(\lambda-2)+a) = 0$$

$$\Rightarrow \underline{\underline{\lambda_1 = 2a}}$$

$$\lambda^2 - 2\lambda + a = 0 \qquad \frac{2 \pm \sqrt{4-4a}}{2} = \frac{2 \pm 2\sqrt{1-a}}{2} = 1 \pm \sqrt{1-a}$$

$$\Rightarrow \underline{\underline{\lambda_2 = 1+\sqrt{1-a}}} \qquad \underline{\underline{\lambda_3 = 1-\sqrt{1-a}}}$$

- $a = 1 \Rightarrow \lambda_1 = 2 \quad \lambda_2 = \lambda_3 = 1$

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & -1 \\ 1 & 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix} \qquad \begin{pmatrix} 2x \\ -z \\ x+y+2z \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix} \Rightarrow \underline{z = -2y}$$

$$x + y - 4y = -4y \qquad \underline{x = -y}$$

$$\Rightarrow \underline{\underline{v_2 = t\begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}}}$$

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & -1 \\ 1 & 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \begin{pmatrix} 2x \\ -z \\ x+y+2z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow x = 0 \quad y = -z$$

$$-z + 2z = z \qquad 0 = 0$$

$$\Rightarrow \underline{\underline{v_1 = t\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}}}$$

- $a = -3 \Rightarrow \lambda_1 = -6 \quad \lambda_2 = 3 \quad \lambda_3 = -1$

$$\begin{pmatrix} -6 & 0 & 0 \\ 0 & 0 & 3 \\ -1 & 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \lambda x \\ \lambda y \\ \lambda z \end{pmatrix} \Rightarrow 3z = -6y \quad \underline{z = -2y}$$

$$-x - y + 2z = -6z \qquad -x - y - 4y = 12y$$

$$17y = -x \qquad \underline{x = -17y}$$

$$\Rightarrow \underline{\underline{v_{-6} = t\begin{pmatrix} -17 \\ 1 \\ -2 \end{pmatrix}}} \qquad \begin{pmatrix} 6 & 0 & 0 \\ 0 & 0 & -3 \end{pmatrix}$$

$$\begin{pmatrix} -6 & 0 & 0 \\ 0 & 0 & 3 \\ -1 & 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \lambda x \\ \lambda y \\ \lambda z \end{pmatrix} \qquad \begin{pmatrix} -6x \\ 3z \\ -x+y+2z \end{pmatrix} = \begin{pmatrix} \lambda x \\ \lambda y \\ \lambda z \end{pmatrix} \Rightarrow \underline{z = \frac{\lambda}{3}y}$$

$$-x + y + \frac{2\lambda}{3}y = \frac{\lambda^2}{3}y \Rightarrow \underline{x = y\left[1 + \frac{2\lambda}{3} - \frac{\lambda^2}{3}\right]}$$

– gives

$-6x = \lambda x \Rightarrow \lambda_1 = -6$ gives nothing here

$$\Rightarrow \underline{V_{-6} = t\begin{pmatrix}-15\\1\\-2\end{pmatrix}}$$

$$\lambda_2 = 3 \Rightarrow x = 0,\ z = y \Rightarrow \underline{V_3 = t\begin{pmatrix}0\\1\\1\end{pmatrix}}$$

$$\lambda_3 = -1 \Rightarrow x = 0,\ z = -\tfrac{1}{3}y \Rightarrow \underline{V_{-1} = t\begin{pmatrix}0\\1\\-\frac{1}{3}\end{pmatrix}}$$

Problem 1-08

$$A = \begin{pmatrix}3 & -1 & 1\\ -1 & 3 & 1\\ 1 & 1 & 3\end{pmatrix}$$

a) $$\begin{vmatrix}3-\lambda & -1 & 1\\ -1 & 3-\lambda & 1\\ 1 & 1 & 3-\lambda\end{vmatrix} = (3-\lambda)\begin{vmatrix}3-\lambda & 1\\ 1 & 3-\lambda\end{vmatrix} + \begin{vmatrix}-1 & 1\\ 1 & 3-\lambda\end{vmatrix} + \begin{vmatrix}-1 & 3-\lambda\\ 1 & 1\end{vmatrix}$$

$$= (3-\lambda)^3 - (3-\lambda) - (3-\lambda) - 1 - 1 - (3-\lambda) = 0 \qquad (*)$$

$$(3-\lambda)^3 = (3-\lambda)(9-6\lambda+\lambda^2) = -\lambda^3+9\lambda^2-27\lambda+27$$

$$(4-\lambda)^3 = (4-\lambda)(16-8\lambda+\lambda^2) = -\lambda^3+12\lambda^2-48\lambda+64$$

$$\Rightarrow (3-\lambda)^3 = (4-\lambda)^3 - 3\lambda^2 + 21\lambda - 37$$

(*) then gives $(4-\lambda)^3 \underbrace{-3\lambda^2+24\lambda-48}_{} = 0$

$(4-\lambda)(3\lambda + x) = -3\lambda^2 - \lambda x + 4x + 12\lambda \Rightarrow x = -12$ does the job

$\Rightarrow$ (*) can be written as $(4-\lambda)\left[(4-\lambda)^2 + (3\lambda-12)\right] = \underline{(4-\lambda)\left[\lambda^2-5\lambda+4\right]}$

$\underline{\lambda_1 = 4 \text{ is one}}$ $\quad \lambda^2 - 5\lambda + 4 = 0 \Rightarrow \lambda = \dfrac{5 \pm \sqrt{25-16}}{2} = \dfrac{5}{2} \pm \dfrac{3}{2}$

$\underline{\lambda_2 = 4} \quad \underline{\lambda_3 = 1}$

b) $$\begin{pmatrix}3 & -1 & 1\\ -1 & 3 & 1\\ 1 & 1 & 3\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt2}\\ 0\\ \frac{1}{\sqrt2}\end{pmatrix} = \begin{pmatrix}\frac{4}{\sqrt2}\\ 0\\ \frac{4}{\sqrt2}\end{pmatrix} = 4\begin{pmatrix}\frac{1}{\sqrt2}\\ 0\\ \frac{1}{\sqrt2}\end{pmatrix}$$

4 is eigenvalue $\Rightarrow \begin{pmatrix}\frac{1}{\sqrt2}\\ 0\\ \frac{1}{\sqrt2}\end{pmatrix}$ is eigenvector

$$\begin{pmatrix}3 & -1 & 1\\ -1 & 3 & 1\\ 1 & 1 & 3\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt6}\\ -\frac{2}{\sqrt6}\\ -\frac{1}{\sqrt6}\end{pmatrix} = \begin{pmatrix}\frac{4}{\sqrt6}\\ -\frac{8}{\sqrt6}\\ -\frac{4}{\sqrt6}\end{pmatrix} = 4\begin{pmatrix}\frac{1}{\sqrt6}\\ -\frac{2}{\sqrt6}\\ -\frac{1}{\sqrt6}\end{pmatrix}$$

$\Rightarrow \begin{pmatrix}\frac{1}{\sqrt6}\\ -\frac{2}{\sqrt6}\\ -\frac{1}{\sqrt6}\end{pmatrix}$ is eigenvector

$$\begin{pmatrix} 3 & -1 & 1 \\ -1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt3} \\ \frac{1}{\sqrt3} \\ -\frac{1}{\sqrt3} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt3} \\ \frac{1}{\sqrt3} \\ -\frac{1}{\sqrt3} \end{pmatrix} \quad \lambda = 1 \text{ is eigenvalue} \quad \Rightarrow \underline{\begin{pmatrix} \frac{1}{\sqrt3} \\ \frac{1}{\sqrt3} \\ -\frac{1}{\sqrt3} \end{pmatrix} \text{ is eigenvector}}$$

c) $$CC' = \begin{pmatrix} \frac{1}{\sqrt2} & \frac{1}{\sqrt6} & \frac{1}{\sqrt3} \\ 0 & -\frac{2}{\sqrt6} & \frac{1}{\sqrt3} \\ \frac{1}{\sqrt2} & -\frac{1}{\sqrt6} & -\frac{1}{\sqrt3} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt2} & 0 & \frac{1}{\sqrt2} \\ \frac{1}{\sqrt6} & -\frac{2}{\sqrt6} & -\frac{1}{\sqrt6} \\ \frac{1}{\sqrt3} & \frac{1}{\sqrt3} & -\frac{1}{\sqrt3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \Rightarrow \underline{CC' = I_3}$$

$$\Rightarrow \underline{C^{-1} = C'}$$

$$C^{-1}AC = C'AC = \begin{pmatrix} \frac{1}{\sqrt2} & 0 & \frac{1}{\sqrt2} \\ \frac{1}{\sqrt6} & -\frac{2}{\sqrt6} & -\frac{1}{\sqrt6} \\ \frac{1}{\sqrt3} & \frac{1}{\sqrt3} & -\frac{1}{\sqrt3} \end{pmatrix}\begin{pmatrix} 3 & -1 & 1 \\ -1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt2} & \frac{1}{\sqrt6} & \frac{1}{\sqrt3} \\ 0 & -\frac{2}{\sqrt6} & \frac{1}{\sqrt3} \\ \frac{1}{\sqrt2} & -\frac{1}{\sqrt6} & -\frac{1}{\sqrt3} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{4}{\sqrt2} & 0 & \frac{4}{\sqrt2} \\ \frac{4}{\sqrt6} & -\frac{8}{\sqrt6} & -\frac{4}{\sqrt6} \\ \frac{1}{\sqrt3} & \frac{1}{\sqrt3} & -\frac{1}{\sqrt3} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt2} & \frac{1}{\sqrt6} & \frac{1}{\sqrt3} \\ 0 & -\frac{2}{\sqrt6} & \frac{1}{\sqrt3} \\ \frac{1}{\sqrt2} & -\frac{1}{\sqrt6} & -\frac{1}{\sqrt3} \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & \underset{1}{4} \end{pmatrix} = \underline{\begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

d) $$D = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix} \qquad B = CDC^{-1} = CDC'$$

$$B^2 = CDC^{-1}CDC^{-1} = \underline{CD^2C^{-1}}$$

$$B^2 = A \quad \Rightarrow \quad CD^2C^{-1} = A \quad \Leftrightarrow$$

$$\Rightarrow \Leftrightarrow D^2 = C^{-1}AC = C'AC = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$D^2 = \begin{pmatrix} d_1^2 & 0 & 0 \\ 0 & d_2^2 & 0 \\ 0 & 0 & d_3^2 \end{pmatrix} \quad \Rightarrow \underline{\text{ok if } d_1^2 = 4,\ d_2^2 = 4,\ d_3^2 = 1}$$

Problem 1-10

a) $|A_k| = \begin{vmatrix} -2 & -1 \\ -1 & 1 \end{vmatrix} - k \begin{vmatrix} 3 & -1 \\ 0 & 1 \end{vmatrix} = -3-3k = -3(1+k) = 0$ if $k=-1$

$\Rightarrow$ $\underline{\underline{k \neq -1 \Rightarrow \text{Rank}(A_k) = 3}}$

$k=-1 \Rightarrow \underline{\underline{\text{Rank}(A_{-1}) = 2}}$

b) $\begin{vmatrix} 1-\lambda & k & 0 \\ 3 & -2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} -2-\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix} - k \begin{vmatrix} 3 & -1 \\ 0 & 1-\lambda \end{vmatrix} = -(1-\lambda)^2(2+\lambda) - (1-\lambda) - 3k(1-\lambda)$

$\underline{\underline{= -(1-\lambda)\left[(2+\lambda)(1-\lambda) + 1 + 3k\right] = 0}}$ $\boxed{\text{Characteristic equation}}$

$\underline{\lambda = 1 \text{ is eigenvalue}}$ $\quad (2+\lambda)(1-\lambda)+1+3k = 0 \quad \Rightarrow \lambda^2 + \lambda - 3(k+1) = 0$

$\Rightarrow \lambda = \dfrac{-1 \pm \sqrt{1+3(k+1)\cdot 4}}{2} = \dfrac{-1 \pm \sqrt{13+12k}}{2}$ $\quad$ $\underline{\underline{\text{real if } 13+12k \geq 0 \text{ i.e. if } k \geq -\tfrac{13}{12}}}$

c) $A_3 = \begin{pmatrix} 1 & 3 & 0 \\ 3 & -2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$ $\Rightarrow \underline{\lambda_1 = 1 \quad \lambda_2 = 3 \quad \lambda_3 = -4}$

$\begin{pmatrix} 1 & 3 & 0 \\ 3 & -2 & -1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{10}} \\ 0 \\ \frac{3}{\sqrt{10}} \end{pmatrix} = \frac{1}{\sqrt{10}}\begin{pmatrix} 1 & 3 & 0 \\ 3 & -2 & -1 \\ 0 & -2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} = \frac{1}{\sqrt{10}}\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$ ok since $\lambda_1 = 1$ is eigenvalue

$\frac{1}{\sqrt{35}}\begin{pmatrix} 1 & 3 & 0 \\ 3 & -2 & -1 \\ 0 & -4 & 1 \end{pmatrix}\begin{pmatrix} -3 \\ 5 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{35}}\begin{pmatrix} 12 \\ -20 \\ -4 \end{pmatrix} = -4\begin{pmatrix} \frac{-3}{\sqrt{35}} \\ \frac{5}{\sqrt{35}} \\ \frac{1}{\sqrt{35}} \end{pmatrix}$ ok since $\lambda_3 = -4$ is eigenvalue.

$\frac{1}{\sqrt{14}}\begin{pmatrix} 1 & 3 & 0 \\ 3 & -2 & -1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{14}}\begin{pmatrix} 9 \\ 6 \\ -3 \end{pmatrix} = 3\begin{pmatrix} \frac{3}{\sqrt{14}} \\ \frac{2}{\sqrt{14}} \\ -\frac{1}{\sqrt{14}} \end{pmatrix}$ ok since $\lambda_2 = 3$ is eigenvalue.

$\Rightarrow$ <u>All three are eigenvectors</u>

They are all three linear independent $\Rightarrow P^{-1}AP = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$

$\underline{\underline{= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -4 \end{pmatrix}}}$

Problem 1-17

a)
$$\begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 1 & 1-\lambda \end{vmatrix} + \begin{vmatrix} 1 & 1-\lambda \\ 1 & 1 \end{vmatrix}$$

$$= (1-\lambda)^3 - (1-\lambda) - (1-\lambda) + 1 + 1 - (1-\lambda) = (1-\lambda)^3 - 3(1-\lambda) + 2$$

$$= (1-\lambda)(1-2\lambda+\lambda^2) - 3(1-\lambda) + 2 = 1 - 2\lambda + \lambda^2 - \lambda + 2\lambda^2 - \lambda^3 - 3 + 3\lambda + 2$$

$$= -\lambda^3 + 3\lambda^2 = 0 \quad \Rightarrow \quad \lambda^2(3-\lambda) = 0 \quad \Rightarrow \quad \underline{\lambda_1 = 0 \quad \lambda_2 = 3}$$

Eigenvectors:
$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} x+y+z \\ x+y+z \\ x+y+z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow x+y+z = 0 \quad x = -y-z$$

$$\underline{v_1 = \begin{pmatrix} -s-t \\ s \\ t \end{pmatrix} \text{ is eigenvector for } \lambda_1 = 0}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3x \\ 3y \\ 3z \end{pmatrix} \quad \begin{pmatrix} x+y+z \\ x+y+z \\ x+y+z \end{pmatrix} = \begin{pmatrix} 3x \\ 3y \\ 3z \end{pmatrix}$$

$$x = \frac{y+z}{2} \qquad \frac{y+z}{2} + y + z = 3y \quad \frac{3z}{2} = \frac{3y}{2} \Rightarrow \underline{y = z}$$

$$\Rightarrow \underline{x = z}$$

$$\Rightarrow \underline{v_2 = t\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \text{ is eigenvector for } \lambda_2 = 3}$$

b)
$$pI_3 + qE = \begin{pmatrix} p+q & q & q \\ q & p+q & q \\ q & q & p+q \end{pmatrix} = \begin{pmatrix} a & b & b \\ b & a & b \\ b & b & a \end{pmatrix} \qquad \underline{b = q \quad a = p+q}$$

c) $A = pI_3 + qE \qquad Ev = \lambda v$

$$Av = (pI_3 + qE)v = pv + qEv = pv + q\lambda v = \underline{(p+q\lambda)v} \Rightarrow \underline{\text{Statement is correct}}$$

d) Eigen value of A equals $p + q\lambda = (a-b) + b\lambda$ where λ is eignvalue of E

$$\Rightarrow \underline{\lambda_1 = a-b} \quad \lambda_2 = a - b + 3b \quad \underline{\lambda_2 = a + 2b}$$

$\boxed{\text{Problem 2-18}}$

a) $$Ax_1 = \begin{pmatrix} -2 & -1 & 4 \\ 2 & 1 & -2 \\ -1 & -1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \underline{\underline{\text{ok}, \lambda_1 = 2}}$$

$$Ax_2 = \begin{pmatrix} -2 & -1 & 4 \\ 2 & 1 & -2 \\ -1 & -1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = -\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \Rightarrow \underline{\underline{\text{ok}, \lambda_2 = -1}}$$

$$Ax_3 = \begin{pmatrix} -2 & -1 & 4 \\ 2 & 1 & -2 \\ -1 & -1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \Rightarrow \underline{\underline{\text{ok}, \lambda_3 = 1}}$$

b) $B = AA$

$$Bx_2 = AAx_2 = A(-1)\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = -Ax_2 = -\left((-1)\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}\right) = \underline{\underline{x_2}}$$

$$Bx_3 = AAx_3 = A\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = Ax_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \underline{\underline{x_3}}$$

$$Bx_1 = AAx_1 = 2Ax_1 = 4x_1 \neq x_1 \qquad \underline{\underline{\text{No}}}$$

c) $C^3 = C^2 + C \qquad Cv = \lambda v$

$C^3 = C^2 + C \;|\cdot v$

$C^2 \lambda v = C\lambda v + \lambda v$

$\Rightarrow \lambda C \lambda v = \lambda^2 v + \lambda v \Rightarrow \lambda^3 v = \lambda^2 v + \lambda v \Rightarrow \underline{\underline{\lambda^3 = \lambda^2 + \lambda}}$

$C + I_n$ has inverse if all eigenvalues unequal zero.

$(C + I_n)v = Cv + I_n v = \lambda_C v + \lambda_I v = (\lambda_C + \lambda_I)v$, that is

the eigenvalues of $C + I_n$ equals eigenvalues of C, λ_C, plus λ_I.

will find λ_I: $I_n v = v \Rightarrow \underline{\lambda_I = 1} \Rightarrow \lambda = \lambda_C + 1$.

Need $\lambda_C \neq -1$; But know $\lambda_C^3 = \lambda_C^2 + \lambda_C$, which is not ok for $\lambda_C = -1 \Rightarrow \underline{\underline{C + I_n \text{ invertible}}}$

Problem 4-01

$$F(T) = \int_0^T f(t) e^{-r(t-T)}\,dt$$

$$F'(T) = f(T) + \int_0^T r f(t) e^{-r(t-T)}\,dt = f(T) + rF(T)$$

$$\Rightarrow F'(T) - rF(T) = f(T)$$

Problem 4-02

$$F(x) = \int_0^x e^{xt^2}\,dt$$

a) $$F'(x) = e^{x^3} + \int_0^x t^2 e^{xt^2}\,dt$$

$$F''(x) = 3x^2 e^{x^3} + x^2 e^{x^3} + \int_0^x t^4 e^{xt^2}\,dt$$

b) $$F''(x) = 4x^2 e^{x^3} + \int_0^x t^4 e^{xt^2}\,dt > 0$$

since $4x^2 e^{x^3} > 0$ $(x>0)$ and $t^4 e^{xt^2} > 0$ for all t from 0 to x.

$\Rightarrow$ F strictly convex

Problem 4-10

$$V(t) = \int_1^t \left(\int_1^t F(t,x,y)\,dx \right) dy$$

$$V'(t) = \int_1^t F(t,x,t)\,dx + \int_1^t \left[F(t,t,y) + \int_1^t \frac{\partial F(t,x,y)}{\partial t}\,dx \right] dy$$

$$= \int_1^t F(t,x,t)\,dx + \int_1^t F(t,t,y)\,dy + \int_1^t\int_1^t \frac{\partial F(t,x,y)}{\partial t}\,dx\,dy$$

Problem 4-11

$$\int_{t_1}^{t} z(\tau)\,d\tau = \int_{t_1+T(t_1)}^{t+T(t)} x(\tau)\,d\tau$$ take the derivative w.r.t. t on both sides

$$z(t) = x(t+T(t))\cdot(1+T'(t))$$

$$\Rightarrow T'(t) = \frac{dT}{dt} = \frac{z(t)}{x(t+T(t))} - 1$$

$$\int_{t}^{t+T(t)} \bar{y}(\tau)\,d\tau = 1$$

$$\Rightarrow \bar{y}(t+T(t))(1+T'(t)) - \bar{y}(t) = 0$$

$$\Rightarrow T'(t) = \frac{dT}{dt} = \frac{\bar{y}(t)}{\bar{y}(t+T(t))} - 1$$