ECON 4140 - Seminar 7

Problem 4-08

$$\int_0^1 \int_0^{\frac{\pi}{2}} x y^2 \cos(x^2 y)\, dy\, dx = \int_0^{\frac{\pi}{2}} \int_0^1 x y^2 \cos(x^2 y)\, dx\, dy$$

$u = x^2 y \quad \frac{du}{dx} = 2xy \quad dx = \frac{1}{2xy} du$

$$= \int_0^{\frac{\pi}{2}} \int_0^{y} \frac{1}{2} y \cos(u)\, du\, dy = \int \left(\Big|_0^{y} \frac{1}{2} y \sin(u) \right) dy = \int_0^{\frac{\pi}{2}} \frac{1}{2} y \sin(y)\, dy$$

$$= \frac{1}{2}\left[\underbrace{\Big|_0^{\frac{\pi}{2}} -y \cos(y)}_{=0} + \int_0^{\frac{\pi}{2}} \cos(y)\, dy \right] =$$

$$= -\frac{1}{2}$$

Problem 4-09

$$\int_{\pi}^{2\pi} \int_0^{\pi} \frac{x}{y^3} \cos\left(\frac{x^2}{y}\right) dx\, dy$$

$u = \frac{x^2}{y} \quad \frac{du}{dx} = 2\frac{x}{y} \quad dx = \frac{1}{2}\frac{y}{x} du$

$$= \int_{\pi}^{2\pi} \int_0^{\frac{\pi^2}{y}} \frac{1}{2}\frac{1}{y^2} \cos(u)\, du\, dy = \int_{\pi}^{2\pi} \left(\Big|_0^{\pi^2} \frac{1}{2}\frac{1}{y^2} \sin(u) \right) dy = \int_{\pi}^{2\pi} \frac{1}{2}\frac{1}{y^2} \sin\left(\frac{\pi^2}{y}\right) dy$$

$v = \frac{\pi^2}{y} \quad \frac{dv}{dy} = -\frac{\pi^2}{y^2} \quad dy = -\frac{y^2}{\pi^2} dv$

$$= \int_{\pi}^{\frac{\pi}{2}} -\frac{1}{2\pi^2} \sin(v)\, dv = \frac{1}{2\pi^2} \int_{\frac{\pi}{2}}^{\pi} \sin(v)\, dv = \frac{1}{2\pi^2} \Big|_{\frac{\pi}{2}}^{\pi} (-\cos v) = \frac{1}{2\pi^2}$$

$\boxed{\text{Problem 6-02}}$

a) $\ddot{x} + \frac{7}{2}\dot{x} - 2x = 0$ $\qquad \frac{1}{4}\left(\frac{7}{2}\right)^2 - (-2) = \frac{49}{16} + 2 > 0$ $\Rightarrow$ Are in case (I)

$\Rightarrow$ $\underline{x = Ae^{r_1 t} + Be^{r_2 t}}$ where $r_{1,2} = -\frac{7}{4} \pm \sqrt{\frac{49}{16} + 2} \Rightarrow \boxed{\begin{matrix} r_1 = \frac{1}{2} \\ r_2 = -4 \end{matrix}}$

b) $f(t) = t + \sin t \Rightarrow u^* = At + B + C\sin t + D\cos t$

$\dot{u}^* = A + C\cos t - D\sin t \qquad \ddot{u}^* = -C\sin t - D\cos t$

$\Rightarrow -C\sin t - D\cos t + \frac{7}{2}(A + C\cos t - D\sin t) - 2(At + B + C\sin t + D\cos t) = t + \sin t$

$\Rightarrow \frac{7}{2}A - 2B = 0 \qquad \underline{B = \frac{7}{4}A} \qquad -2A = 1 \Rightarrow \boxed{A = -\frac{1}{2}} \Rightarrow \boxed{B = -\frac{7}{8}}$

$-D + \frac{7}{2}C - 2D = 0 \Rightarrow \underline{D = \frac{7}{6}C}$

$-C - \frac{7}{2}D - 2C = 1 \qquad \Rightarrow -3C - \frac{7}{2}\frac{7}{6}C = 1 \qquad C\left[3 + \frac{49}{12}\right] = -1 \qquad \boxed{C = -\frac{12}{61}}$

$\boxed{D = -\frac{14}{61}}$ $\qquad \underline{u^* = -\frac{1}{2}t - \frac{7}{8} - \frac{12}{61}\sin t - \frac{14}{61}\cos t}$

$\Rightarrow \quad \underline{\underline{x = Ae^{\frac{1}{2}t} + Be^{-4t} - \frac{1}{2}t - \frac{7}{8} - \frac{12}{61}\sin t - \frac{14}{61}\cos t}}$

$\boxed{\text{Problem 6-11}}$

$$\ddot{x} - 2(k-1)\dot{x} + (k^2-4)x = 2e^{(4-k)t}$$

a) $\ddot{x} - 2(k-1)\dot{x} + (k^2-4)x = 0$

$$\frac{1}{4}(-2(k-1))^2 - (k^2-4) = (k-1)^2 - k^2 + 4 = \underline{5-2k} \begin{cases} >0 & \text{if } k<\frac{5}{2} \\ =0 & \text{if } k=\frac{5}{2} \\ <0 & \text{if } k>\frac{5}{2} \end{cases}$$

• $k<\frac{5}{2} \Rightarrow \underline{\underline{x = Ae^{r_1 t} + Be^{r_2 t}, \quad r_{1,2} = (k-1) \pm \sqrt{5-2k}}}$

• $k=\frac{5}{2} \Rightarrow \underline{\underline{x = (A+Bt)e^{rt}, \quad r = k-1}}$

• $k<\frac{5}{2} \Rightarrow \underline{\underline{x = e^{(k-1)t}(A\cos\beta t + B\sin\beta t), \quad \beta = \sqrt{2k-5}}}$

b) $f(t) = 2e^{(4-k)t} \Rightarrow u^* = \frac{2}{(4-k)^2 - 2(k-1)(4-k) + (k^2-4)} e^{(4-k)t} = \underline{\underline{\frac{2e^{(4-k)t}}{4k^2 - 18k + 20}}}$

c) $k=3 \Rightarrow \underline{x = e^{2t}(A\cos t + B\sin t) + e^{t}}$

Passes through origin $\Rightarrow x(0) = 0 \Rightarrow A + 1 = 0 \Rightarrow \underline{A = -1}$

Tangent to t-axis at origin $\Rightarrow \dot{x}(0) = 0$

$\dot{x} = 2e^{2t}(B\sin t - \cos t) + e^{2t}(B\cos t + \sin t) + e^{t}$

$\dot{x}(0) = 2(-1) + B + 1 = 0 \Rightarrow \underline{B = 1}$

$\Rightarrow \underline{\underline{x = e^{2t}(\sin t - \cos t) + e^{t}}}$

$\dot{x} = 2e^{2t}(\sin t - \cos t) + e^{2t}(\cos t + \sin t) + e^{t} = e^{2t}(3\sin t - \cos t) +$

$\ddot{x} = 2e^{2t}(3\sin t - \cos t) + e^{2t}(3\cos t + \sin t) + e^{t} = e^{2t}(7\sin t + \cos t) + e^{t}$

$\ddot{x}(0) = 1 + 1 = 2 > 0 \Rightarrow$ $\underline{\underline{(x,t) = (0,0) \text{ is local minimum, and thus local extrema}}}$

$\boxed{\text{Problem 6-15}}$

a) $2\ddot{x} + 8\dot{x} + 26x = e^{2t} \Rightarrow \ddot{x} + 4\dot{x} + 13x = \frac{1}{2}e^{2t}$

Homogeneous: $\ddot{x} + 4\dot{x} + 13x = 0 \quad \frac{1}{4}4^2 - 13 = 4 - 13 < 0$

$\Rightarrow$ ~~$x = e^{-2t}(A\cos 3t$~~ $\underline{x = e^{-2t}(A\cos(3t) + B\sin(3t))}$

$$u^* = \frac{\frac{1}{2}}{4 + \frac{1}{2}\cdot 4 + 13} e^{2t} = \frac{e^{2t}}{8 + 4 + 26} = \underline{\frac{e^{2t}}{38}}$$

$$\Rightarrow \underline{\underline{x = e^{-2t}(A\cos(3t) + B\sin(3t)) + \frac{e^{2t}}{38}}}$$

b) e^{2t} replaced by $\sin 3t \Rightarrow \ddot{x} + 4\dot{x} + 13x = \frac{1}{2}\sin 3t \Rightarrow f(t) = \frac{1}{2}\sin 3t$

$\Rightarrow u^* = A\sin 3t + B\cos 3t$, where A and B is found by fitting the relevant coefficients in the equation $\ddot{u}^* + 4\dot{u}^* + 13u^* = \frac{1}{2}\sin 3t$.

$\boxed{\text{Problem 6-10}}\boxed{\text{b\&c}}$

b) $\frac{1}{2}\sigma^2 x^2 V''(x) + \mu x V'(x) - \rho V(x) = w - x$

A particular solution: $u^* = Ax + B \quad \dot{u}^* = A \quad \ddot{u}^* = 0$

$\Rightarrow \mu x A - \rho(Ax + B) = w - x \Rightarrow \mu A - \rho A = -1 \quad \boxed{A = \frac{1}{\rho - \mu}}$

$-\rho B = w \quad \boxed{B = -\frac{w}{\rho}} \quad \Rightarrow \underline{\underline{u^* = \frac{x}{\rho - \mu} - \frac{w}{\rho}}}$

c) Here, $\sigma = \sqrt{2}, \mu = 1, \rho = 4, w = 10$.

$\Rightarrow$ ~~$V(x) = Ax^a + Bx^b + \frac{1}{3}x - \frac{5}{2}$~~

Where a, b is solutions to $a(a-1) + a - 4 = 0 \quad a^2 - 4 = 0 \Rightarrow a = \pm 2$

$$\Rightarrow \underline{\underline{V(x) = Ax^2 + Bx^{-2} + \frac{1}{3}x - \frac{5}{2}}}$$

$\boxed{\text{Problem 6-13}}$

$$t^2\ddot{x} + t\dot{x} - x = 0 \quad , t>0$$

a) (i) Without substitution:

Characteristic equation $r^2 - 1 = 0 \Rightarrow r^2 = 1 \Rightarrow \underline{r_{1,2} = \pm 1}$

$$\Rightarrow \underline{\underline{x = At + Bt^{-1}}}$$

(ii) With substitution:

$z = tx \qquad \dot{z} = x + t\dot{x} \qquad \ddot{z} = \dot{x} + \dot{x} + t\ddot{x} = 2\dot{x} + t\ddot{x}$

$$\Rightarrow t^2\ddot{x} + t\dot{x} - x = t\ddot{z} - \dot{z}$$

So get $\ddot{z} - \frac{1}{t}\dot{z} = 0 \qquad$ Let $y = \dot{z}$

$\Rightarrow \dot{y} - \frac{1}{t}y = 0 \qquad \frac{1}{y}dy = \frac{1}{t}dt \qquad \ln y = \ln t + C \qquad y = te^C \quad \underline{y = tD}$

$\Rightarrow \dot{z} = tD \qquad z = \frac{1}{2}t^2D + E \qquad z = D_2 t^2 + E = tx$

$$\Rightarrow \underline{\underline{x = D_2 t + Et^{-1}}}$$

b) $x(1) = A + B = 1 \qquad \underline{B = 1 - A}$

$\dot{x} = A - Bt^{-2} \qquad \dot{x}(1) = A - B = A + A - 1 = 1 \Rightarrow \underline{A = 1} \Rightarrow \underline{B = 0}$

$$\Rightarrow \underline{\underline{x = t}}$$

Problem 6-12

$$\dot{p}(t) = \beta \int_{-\infty}^{t} [a+c-(b+d)p(\tau)]\, e^{-\gamma(t-\tau)}\, d\tau$$

a) $\ddot{p} = \beta\left[a+c-(b+d)p(t) - \gamma\int_{-\infty}^{t}(a+c-(b+d)p(\tau))\, e^{-\gamma(t-\tau)}\, d\tau\right]$

$= \beta(a+c) - \beta(b+d)p - \gamma\dot{p}$

$\Rightarrow$ $\underline{\ddot{p} + \gamma\dot{p} + \beta(b+d)p = \beta(a+c)}$

b) $\dot{p} = 0$ and $\ddot{p} = 0$ $\Rightarrow$ $\underline{p^* = \dfrac{a+c}{b+d}}$

General solution: $\frac{1}{4}\gamma^2 - \beta(b+d) \begin{cases} >0 & \text{case I} \\ =0 & \text{case II} \\ <0 & \text{case III} \end{cases}$

Case I: $p = Ae^{r_1 t} + Be^{r_2 t}$ where $r_{1,2} = -\frac{1}{2}\gamma \pm \sqrt{\frac{1}{4}\gamma^2 - \beta(b+d)}$

case II: $p = (A+Bt)e^{-\frac{1}{2}\gamma t}$

Case III: $p = e^{-\frac{1}{2}\gamma t}(A\cos \phi t + B\sin \phi t)$ where $\phi = \sqrt{\beta(b+d) - \frac{1}{4}\gamma^2}$

To these general solution is $p + v^*$, when $v^* = p^*$ $\Rightarrow$ $p + \dfrac{a+c}{b+d}$

c) Since $\gamma > 0$, $-\frac{1}{2}\gamma < 0$, and also note that both $r_1 < 0$ and $r_2 < 0$

Thus, case I $\lim_{t\to\infty}(p + p^*) = p^*$ since $\lim_{t\to\infty} p = 0$

case II $\lim_{t\to\infty} (A+Bt)e^{-\frac{1}{2}\gamma t} + p^* = p^*$ (even though $Bt \to \infty$, the exponent will dominate!

case III $\lim_{t\to\infty}(p + p^*) = p^*$ ~~fluctuate in the interval~~

$\Rightarrow$ Stable in all cases

In the case $\frac{1}{4}\gamma^2 - \beta(b+d) < 0$ we have the fluctuating case due to sin, cos functions.

Problem S-06

$$\dot K = \gamma Q \qquad Q = K^{\gamma} L \qquad \dot L = \beta \qquad \gamma, \beta > 0 \qquad 0 < \gamma < 1$$

a) $$\dot K = \gamma Q = \gamma K^{\gamma} L \qquad \dot L = \beta \qquad \frac{dL}{dt} = \beta \qquad L(t) = \beta t + C$$

$$\Rightarrow \quad \dot K = \gamma K^{\gamma}(\beta t + C)$$

b) $$L(0) = L_0 \;\Rightarrow\; C = L_0 \;\Rightarrow\; L = \beta t + L_0$$

$$\Rightarrow \quad \dot K = \gamma K^{\gamma}(\beta t + L_0) \;\Rightarrow\; \int \frac{1}{\gamma K^{\gamma}}\, dK = \int \beta t + L_0 \, dt$$

$$\frac{1}{\gamma} K^{1-\gamma} \frac{1}{1-\gamma} = \frac{1}{2}\beta t^2 + L_0 t + C \qquad K = \left[\frac{1}{2}\beta t^2 + L_0 t + C\right]^{\frac{1}{1-\gamma}} (1-\gamma)^{\frac{1}{1-\gamma}} \gamma^{\frac{1}{1-\gamma}}$$

$$K(0) = \left[C(1-\gamma)\gamma\right]^{\frac{1}{1-\gamma}} = K_0 \qquad C = K_0^{1-\gamma} \frac{1}{(1-\gamma)\gamma}$$

$$\Rightarrow \quad K = \left[\frac{1}{2}\beta t^2 + L_0 t + \frac{K_0^{1-\gamma}}{(1-\gamma)\gamma}\right]^{\frac{1}{1-\gamma}} \left[(1-\gamma)\gamma\right]^{\frac{1}{1-\gamma}}$$

Problem S-11

$$e^{2t}\dot x + e^{2t}(2-2t)x = \frac{e^{t^2+t}}{\sqrt{1+e^t}} \qquad e^{2t-t^2}\dot x + e^{2t-t^2}(2-2t)x = \frac{e^t}{\sqrt{1+e^t}}$$

$$\frac{d}{dt}\left(e^{2t-t^2}x\right) = \frac{e^t}{\sqrt{1+e^t}} \qquad e^{2t-t^2}x = \int \frac{e^t}{\sqrt{1+e^t}}\, dt \qquad u = e^t \quad \frac{du}{dt} = e^t \quad dt = \frac{1}{e^t}\, du$$

$$\Rightarrow \quad e^{2t-t^2}x = \int \frac{1}{\sqrt{1+u}}\, du = 2(1+u)^{\frac{1}{2}} + C = 2(1+e^t)^{\frac{1}{2}} + C$$

$$x = 2\sqrt{1+e^t}\, e^{t^2-2t} + C e^{t^2-2t} \qquad x(-1) = 2\sqrt{1+e^{-1}}\, e^{3} + C e^{3} = 0 \qquad C = -2\sqrt{1+e^{-1}}$$

$$\Rightarrow \quad x = 2\sqrt{1+e^t}\, e^{t^2-2t} - 2\sqrt{1+e^{-1}}\, e^{t^2-2t} \qquad x = 2e^{t^2-2t}\left[\sqrt{1+e^t} - \sqrt{1+e^{-1}}\right]$$

Problem 5-14

$\dot{x} = x^3 + 3x^2 - 2 \qquad x = y + a \Rightarrow \dot{x} = \dot{y}\ ; \ x^3 = (y+a)(y^2+2ya+a^2) = y^3 + 3ay^2 + 3a^2y + a^3$

$x^2 = y^2 + 2ya + a^2$

$\Rightarrow \dot{y} = y^3 + 3(a+1)y^2 + 3a[a+2]y + a^2[3+a] - 2$

$a = -1$ gives $\underline{\dot{y} + 3y = y^3}$ Bernoulli's equation

$z = y^{1-3} \quad \underline{z = y^{-2}} \qquad -\frac{1}{2}\dot{z} + 3z = 1 \Rightarrow \dot{z} - 6z = -2$

$\frac{d}{dt}(ze^{-6t}) = -2e^{-6t} \quad ze^{-6t} = \frac{1}{3}e^{-6t} + C \qquad \underline{z = \frac{1}{3} + Ce^{6t}}$

$\Rightarrow y = (-2te^{6t} + Ce^{6t})^{-\frac{1}{2}} = \underline{\underline{(C-2t)^{-\frac{1}{2}}e^{-3t}}} \qquad y = \left(\frac{1}{3} + Ce^{6t}\right)^{-\frac{1}{2}}$

$\underline{\underline{x = \left(\frac{1}{3} + Ce^{6t}\right)^{-\frac{1}{2}} - 1}}$

Problem 6-08

i) $g''(t) = -\frac{1}{4}g'(t)$ Define $y = g'(t)$

The equation then says $\dot{y} = -\frac{1}{4}y \Rightarrow \dot{y} + \frac{1}{4}y = 0 \quad \frac{d}{dt}(ye^{\frac{1}{4}t}) = 0 \quad y = Ce^{-\frac{1}{4}t}$

$\Rightarrow \underline{\underline{g(t) = -4Ce^{-\frac{1}{4}t} + D}}$

ii) $g''(t) = -\frac{2}{t+1}g'(t)$ Define $y = g'(t)$

The equation then says $\dot{y} = -\frac{2}{t+1}y \qquad \dot{y} + \frac{2}{t+1}y = 0$

$\frac{d}{dt}\left(ye^{2\ln(t+1)}\right) = 0 \qquad y(t+1)^2 = C \qquad y = \frac{C}{(t+1)^2}$

$\Rightarrow \underline{\underline{g(t) = -\frac{C}{t+1} + D}}$

Problem 6-10a

a) $\frac{1}{2}\sigma^2x^2V''(x) + \mu xV'(x) - \rho V(x) = w - x \qquad V(x) = Ax^a + Bx^b$

$V'(x) = aAx^{a-1} + bBx^{b-1} \qquad V''(x) = a(a-1)Ax^{a-2} + b(b-1)Bx^{b-2}$

$\Rightarrow \frac{1}{2}\sigma^2[a(a-1)Ax^a + b(b-1)x^b] + \mu[aAx^a + bBx^b] - \rho[Ax^a + Bx^b] = 0 \qquad (*)$

$\Rightarrow \frac{1}{2}\sigma^2a(a-1) + \mu a - \rho = 0$ Gives a and b solutions to this equation, and $V(x) = Ax^a + Bx^b$, A, B general constants.

$\boxed{6-01}$

a) $\ddot{x} - 8\dot{x} + 17 = 0 \Rightarrow \ddot{x} - 8\dot{x} = -17$

Start with the homogeneous version

$\ddot{x} - 8\dot{x} = 0.$

The characteristic equation has two real roots, and the solution is

$\underline{x(t) = Ae^{8t} + B}$.

Now lets examine the general solution

$x(t) = Ae^{8t} + B + u^*(t).$

When $f(t)$ is a constant, but there is now pure x term in the problem $(b = 0)$, it is reasonable to have a linear expression $u^*(t) = \alpha t + \beta$ as particular solution.

We then get $u^{*\prime\prime}(t) - 8u^{*\prime}(t) = -17$

$0 - 8\alpha = -17 \Rightarrow \underline{\alpha = \frac{17}{8}}$, β may be any number, and we define $C_2 := C_1 + \beta$.

Then the general solution is

$\underline{\underline{x(t) = Ae^{8t} + \frac{17}{8}t + C_2}}$

To solve this exercise, you could also have integrated on both side of the problem to get an ordinary 1. order differential equation.

b) $\ddot{x} + 2\dot{x} + 5x = 0$

First we examine the characteristic equation:

$\frac{1}{4}(2)^2 - 5 = 1 - 5 = -4 < 0$

$\Rightarrow$ no real solutions.

Thus, the solution to this problem is

$x(t) = e^{\alpha t}(A\cos\beta t + B\sin\beta t)$, where $\alpha = -\frac{1}{2} \cdot 2 = -1$, $\beta = \sqrt{5-1} = 2$.

Thus, $\underline{\underline{x(t) = e^{-t}[A\cos(2t) + B\sin(2t)]}}$