ECON 4140 - Seminar 9

$\boxed{\text{Problem 10-3}}$

$$x_{t+2} + x_{t+1} - 6x_t = 5^t + t$$

$$1^2 + 6 \cdot 4 > 0 \quad \Rightarrow \quad x_t = A m_1^t + B m_2^t \qquad m_{1,2} = -\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{25} = -\tfrac{1}{2} \pm \tfrac{5}{2}$$

$$m_1 = 2 \qquad m_2 = -3$$

$$u_t = C5^t + Dt + E$$

$$C5^{t+2} + D(t+2) + E + C5^{t+1} + D(t+1) + E - 6(C5^t + Dt + E) = 5^t + t$$

$$5^t(25C + 5C - 6C) + t(D + D - 6D) + 2D + E + D + E - 6E = 5^t + t$$

$24C = 1$ $\quad \underline{C = \frac{1}{24}}$ $\quad -4D = 1$ $\quad \underline{D = -\frac{1}{4}}$ $\quad -\frac{3}{4} - 4E = 0$ $\quad \underline{E = -\frac{3}{16}}$

$$\Rightarrow \quad \underline{\underline{x_t = A2^t + B(-3)^t + \frac{1}{24}5^t - \frac{1}{4}t - \frac{3}{16}}}$$

$\boxed{10-04}$

$x_{t+2} + 4x_{t+1} - 12x_t = 7t^2 + 2t - 6 \qquad x_0 = -3 \quad x_1 = 9$

$4^2 - 4(-12) > 0 \Rightarrow x_t = Am_1^t + Bm_2^t \quad m_{1,2} = -2 \pm \frac{1}{2}\sqrt{64} = -2 \pm 4 \quad \underline{m_1 = 2} \; \underline{m_2 = -6}$

$u_t = Ct^2 + Dt + E$

$\cancel{C(t+2)^2 + D(t+2) + E + C(t+1)^2 + D(t+1) + E + Ct^2 + Dt + E}$

$C(t+2)^2 + D(t+2) + E + 4[C(t+1)^2 + D(t+1) + E] - 12[Ct^2 + Dt + E] = 7t^2 + 2t - 6$

$t^2[C + 4C - 12C] + t[4C + D + 8C + 4D - 12D] + [4C + 2D + E + 4C + 4D + 4E - 12E] = 7t^2 + 2t - 6$

$-7C = 7 \quad \underline{C = -1} \qquad -12 - 7D = 2 \quad \underline{D = -2}$

$-8 - 12 - 7E = -6 \quad \underline{E = -2}$

$\Rightarrow \underline{x_t = A2^t + B(-6)^t - t^2 - 2t - 2}$

$x_0 = A + B - 2 = -3 \quad \underline{A = -1 - B}$

$x_1 = -(1+B)2 - 6B - 1 - 2 - 2 = 9 \quad 8B = -16 \quad \underline{B = -2} \quad \underline{A = 1}$

$\Rightarrow \underline{\underline{x_t = 2^t - 2(-6)^t - t^2 - 2t - 2}}$

$\boxed{10-05}$

a) $x_{t+2} - \frac{5}{2}x_{t+1} + x_t = 10\cdot 3^t$ $\quad x_0 = 0 \quad x_1 = 2$

$\frac{25}{4} - 4 = \frac{25-16}{4} = \frac{9}{4} > 0 \Rightarrow m_{1,2} = \frac{5}{4} \pm \frac{1}{2}\sqrt{\frac{9}{4}} = \frac{5}{4} \pm \frac{3}{4}$ $\quad \underline{m_1 = 2} \quad \underline{m_2 = \frac{1}{2}}$

$\underline{x_t = A2^t + B2^{-t}}$

$u_t = C3^t$

$C3^{t+2} - \frac{5}{2}C3^{t+1} + C3^t = 10\cdot 3^t$

$3^t\left[9C - \frac{15}{2}C + C\right] = 10\cdot 3^t$ $\quad \frac{18-15+2}{2}C = 10$ $\quad C = \frac{20}{5}$ $\quad \underline{C = 4}$

$\Rightarrow \underline{x_t = A2^t + B2^{-t} + 4\cdot 3^t}$

$x_0 = A + B + 4 = 0$ $\quad A = -B - 4$

$x_1 = -2B - 8 + \frac{1}{2}B + 12 = 2$ $\quad \frac{3}{2}B = 2$ $\quad \underline{B = \frac{4}{3}}$ $\quad A = -\frac{4}{3} - 4 = \underline{-\frac{16}{3}}$

$$\underline{\underline{x_t = -\frac{16}{3}2^t + \frac{4}{3}2^{-t} + 4\cdot 3^t}}$$

b) $-\alpha x_{t+1} + (1+\alpha^2)x_t - \alpha x_{t-1} = K\beta^t$ $\quad \alpha, \beta > 0, \ \alpha \neq 1, \beta \neq \alpha, \beta \neq \frac{1}{\alpha}$

$x_{t+1} - \frac{1+\alpha^2}{\alpha}x_t + x_{t-1} = -\frac{K}{\alpha}\beta^t = -\frac{K}{\alpha}\beta\beta^{t-1}$

$\Rightarrow \boxed{x_{t+2} - \frac{1+\alpha^2}{\alpha}x_{t+1} + x_t = -\frac{\beta}{\alpha}K\beta^t}$ $\quad \frac{(1+\alpha^2)^2}{\alpha^2} - 4 = \frac{1}{\alpha^2} + 2 + \alpha^2 - 4 = \alpha^2 + \frac{1}{\alpha^2} - 2$

$\circ > 0$ if $\alpha^2 + \frac{1}{\alpha^2} - 2 > 0$ $\quad \alpha^4 - 2\alpha^2 + 1 > 0$ $\quad a^2 - 2a + 1 > 0$ when $a = \alpha^2$

i.e. $(a-1)^2 > 0$ will always happen since $\alpha \neq 1$.

Thus we are in Case (I) $\Rightarrow x_t = Am_1^t + Bm_2^t$, $m_{1,2} = \frac{1+\alpha^2}{2\alpha} \pm \frac{1}{2}\sqrt{\alpha^2 + \frac{1}{\alpha^2} - 2} = \frac{1+\alpha^2}{2\alpha} \pm \frac{1}{2}\sqrt{\frac{(\alpha^2-1)^2}{\alpha^2}}$

$m_{1,2} = \frac{1+\alpha^2}{2\alpha} \pm \frac{(\alpha^2-1)}{2\alpha}$ $\quad \underline{m_1 = \alpha} \quad \underline{m_2 = \alpha^{-1}} \quad \Rightarrow \underline{x_t = A\alpha^t + B\alpha^{-t}}$

$y_t = C\beta^t$

$C\beta^{t+2} - \frac{1+\alpha^2}{\alpha} C\beta^{t+1} + C\beta^t = -\frac{\beta}{\alpha} K \beta^t$

$C\left[\beta^2 - \frac{1+\alpha^2}{\alpha}\beta + 1\right] = -\frac{\beta}{\alpha} K \quad \underline{C = \frac{-\beta K}{\alpha\beta^2 - (1+\alpha^2)\beta + \alpha}}$

$$\Rightarrow x_t = A\alpha^t + B\alpha^{-t} - \frac{K}{\alpha\beta^2 - (1+\alpha^2)\beta + \alpha}\beta^{t+1}$$

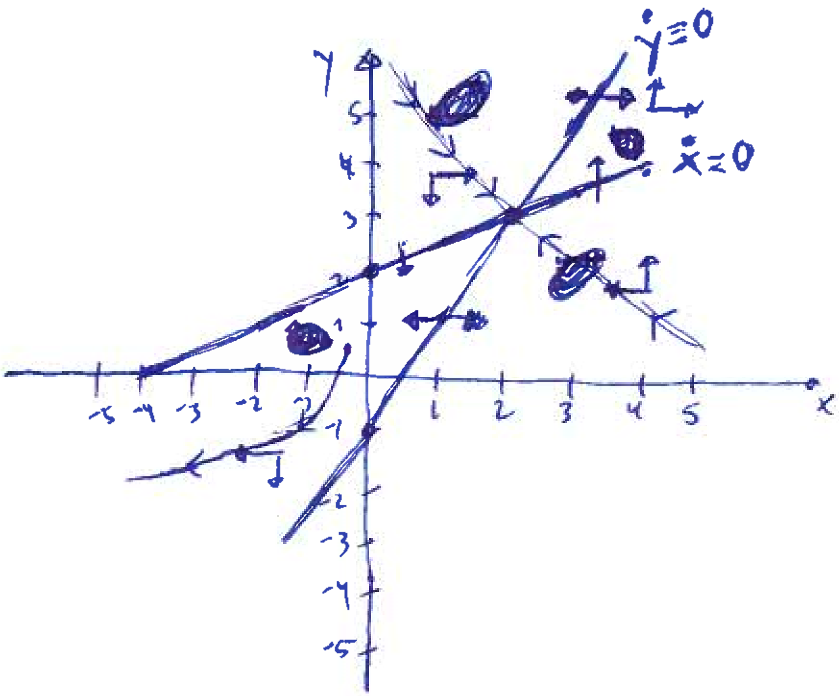
$\boxed{7\text{-}02\ c}$

$a = -1, \quad \alpha = -4, \quad \beta = -1$

$$\dot{x} = -x + 2y - 4$$
$$\dot{y} = 2x - y - 1$$

$\dot{x} = 0 \iff y = \frac{1}{2}x + 2 \quad , \quad \dot{x} > 0 \iff y > \frac{1}{2}x + 2$

$\dot{y} = 0 \iff y = 2x - 1 \quad , \quad \dot{y} > 0 \iff y < 2x - 1$



For instance letting $A = 0$ and $B = 1$ of the solution from a) will make the solution converge.

Thus, $x(t) = e^{-3t} + 2$, $y(t) = -e^{-3t} + 2$ is a solution curve

$\boxed{\text{Problem 7-04}}\ \boxed{a \,\&\, b}$

$$\dot{x} = y - x^2 - xy = f(x,y)$$
$$\dot{y} = x - y^2 - xy = g(x,y)$$

a) $\dot{x} = 0 \Rightarrow y - x^2 - xy = 0 \Rightarrow xy = y - x^2$

$\dot{y} = 0 \Rightarrow x - y^2 - xy = 0$

$x - y^2 - y + x^2 = 0 \qquad x + x^2 = y + y^2 \Rightarrow \underline{x = y}$

$\Rightarrow x - 2x^2 = 0 \qquad x^2 - \frac{1}{2}x = 0 \qquad x(x - \frac{1}{2}) = 0 \qquad x = 0 \text{ or } x = \frac{1}{2}$

$\Rightarrow \boxed{(x,y) = (0,0) \text{ or } (x,y) = (\frac{1}{2}, \frac{1}{2})}$

~~0 = y + x^2 - xy partial derivative w.r.t. x g(x) 0 = 2x - y~~

$f_1' = -2x - y \qquad f_2' = 1 - x$

$g_1' = 1 - y \qquad g_2' = -2y - x$

$A(0,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow tr(A(0,0)) = 0,\ |A(0,0)| = -1 < 0$

Since the determinant is negative, $\underline{(0,0) \text{ is saddle point}}$

$A(\frac{1}{2},\frac{1}{2}) = \begin{pmatrix} -\frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{3}{2} \end{pmatrix} \Rightarrow tr(A(\frac{1}{2},\frac{1}{2})) = -3 < 0 \quad |A(\frac{1}{2},\frac{1}{2})| = \frac{9}{4} - \frac{1}{4} = 2 > 0$

Since determinant positive and trace negative, $\underline{(\frac{1}{2},\frac{1}{2}) \text{ is asymptotically stable}}$

b) $z = x + y \quad \dot{z} = \dot{x} + \dot{y} = y - x^2 - xy + x - y^2 - xy = x + y - (x^2 + 2xy + y^2) = z - z^2$

$\Rightarrow \underline{\dot{z} = z - z^2}$ ~~Bernoulli form~~. i.e. $\underline{\dot{z} - z = -z^2}$ Bernoulli form

$\Rightarrow w = z^{-1} \qquad -\dot{w} - w = -1 \qquad \dot{w} + w = 1 \quad \frac{d}{dt}(e^t w) = e^t \quad e^t w = e^t + C$

~~$w = z^{-1} = 1 + Ce^{-t} \Rightarrow z = \frac{1}{1+Ce^{-t}}$~~ $w = 1 + Ce^{-t} \Rightarrow z = \frac{1}{1+Ce^{-t}} \quad \underline{z = \frac{e^t}{e^t + C}}$

$\boxed{7\text{-}04_c}$

Notice from $(*)$ that when $x = y$, then $\dot{x} = \dot{y}$ must hold, and from this we deduce that if $x(t) = y(t)$ for one point t, then $x(t) = y(t)$ for all t.

Thus, we let $x = y$, and $(*)$ reduces to

$$\dot{x} = x - 2x^2 \quad (\text{and } \dot{y} = y - 2y^2).$$

This is a Bernoulli equation : $\boxed{\dot{x} - x = -2x^2}$

$$z = x^{-1} \quad \Rightarrow \quad -\dot{z} - z = -2 \qquad \dot{z} + z = 2 \quad \Rightarrow \quad \underline{z = Ce^{-t} + 2}$$

$$\Rightarrow \quad x = \frac{1}{Ce^{-t} + 2} \quad \Rightarrow \quad \boxed{x = \frac{e^t}{2e^t + C} = y}$$

- For (x, y) to pass through $(1,1)$, we can for instance let this happen at $t = 0$, $\Rightarrow \quad \frac{1}{2+C} = 1 \quad \Rightarrow \quad C = -1$

$$\Rightarrow \quad \underline{\underline{x(t) = y(t) = \frac{e^t}{2e^t - 1}}}$$

- For (x,y) through $(\frac{1}{4}, \frac{1}{4})$, we get $\frac{1}{2+C} = \frac{1}{4} \qquad \frac{1}{2} + \frac{1}{4}C = 1$

$$C = 2 \qquad \Rightarrow \quad \underline{\underline{x(t) = y(t) = \frac{e^t}{2(e^t + 1)}}}$$

- For (x,y) through $(-1,-1)$ we get $\frac{1}{2+C} = -1 \qquad 1 = -2 - C \qquad C = -3$

$$\Rightarrow \quad \underline{\underline{x(t) = y(t) = \frac{e^t}{2e^t - 3}}}$$

$\boxed{\text{Exercise by Fromstad}}$

$\dot{x} = 1 - e^{x-y} \qquad \dot{y} = -y$

$\dot{x} = 0 \Rightarrow 1 - e^{x-y} = 0$

$\dot{y} = 0 \Rightarrow -y = 0 \Rightarrow \boxed{y=0} \Rightarrow 1 - e^{x} = 0 \Rightarrow e^{x} = 1 \Rightarrow \boxed{x=0}$

$\Rightarrow \underline{(x,y) = (0,0) \text{ is equilibrium point}}$ $\quad A(x,y) = \begin{pmatrix} -e^{x-y} & e^{x-y} \\ 0 & -1 \end{pmatrix}$ Easy to verify the conditions of Olech's theorem.

$\boxed{\text{Exercise 7-05}}$

$\dot{x} = \underbrace{\tfrac{1}{2}x^3 - y}_{f(x,y)} \qquad \dot{y} = \underbrace{2x - y}_{g(x,y)}$

a) $\dot{x} = 0 \Rightarrow \tfrac{1}{2}x^3 - y = 0 \quad \underline{y = \tfrac{1}{2}x^3}$

$\dot{y} = 0 \Rightarrow 2x - y = 0 \quad 2x = \tfrac{1}{2}x^3 \quad \underline{x=0}$ or $x^2 = 4$ is $\underline{x = \pm 2}$

$\Rightarrow y = 0$ or $y = \pm 4$

$\boxed{(x,y) = (0,0),\ (2,4),\ (-2,-4)}$ Equilibrium points.

$f_x' = \tfrac{3}{2}x^2 \quad f_y' = -1 \qquad A = \begin{pmatrix} \tfrac{3}{2}x^2 & -1 \\ 2 & -1 \end{pmatrix} \qquad tr(A) = \tfrac{3}{2}x^2 - 1$

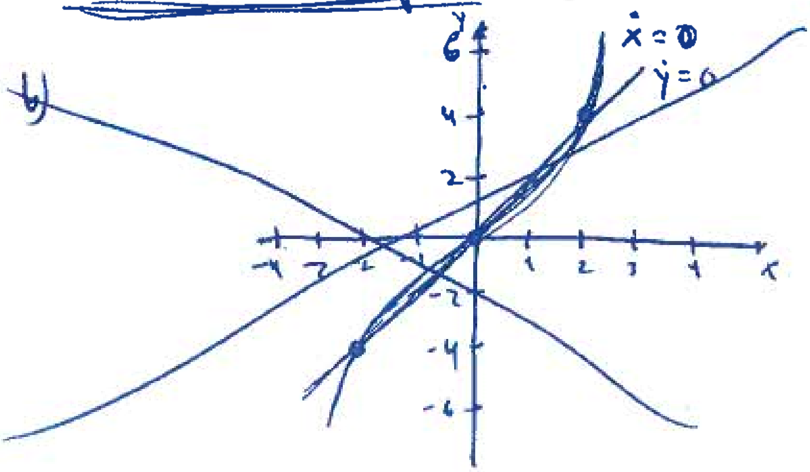
$g_x' = 2 \quad g_y' = -1 \qquad\qquad |A| = -\tfrac{3}{2}x^2 + 2$

$(0,0)$: $tr(A) < 0 \quad |A| > 0 \Rightarrow$ Asymptotically stable

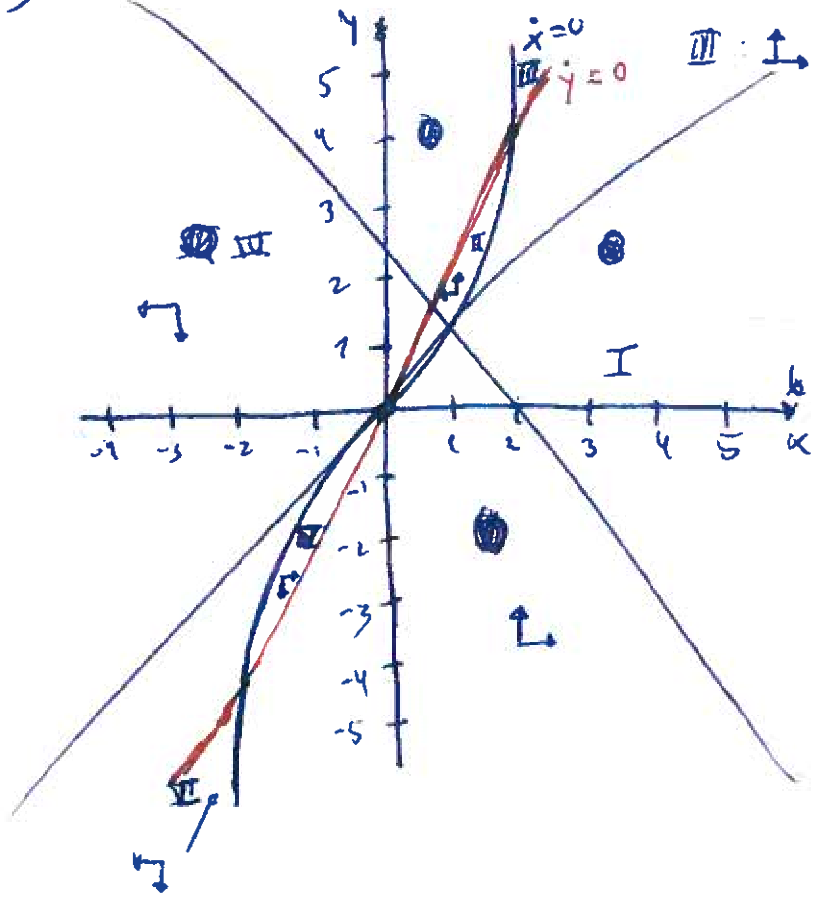
$(2,4)$: $tr(A) > 0 \quad |A| < 0 \Rightarrow$ Saddle path

$(-2,-4)$ Also saddle path

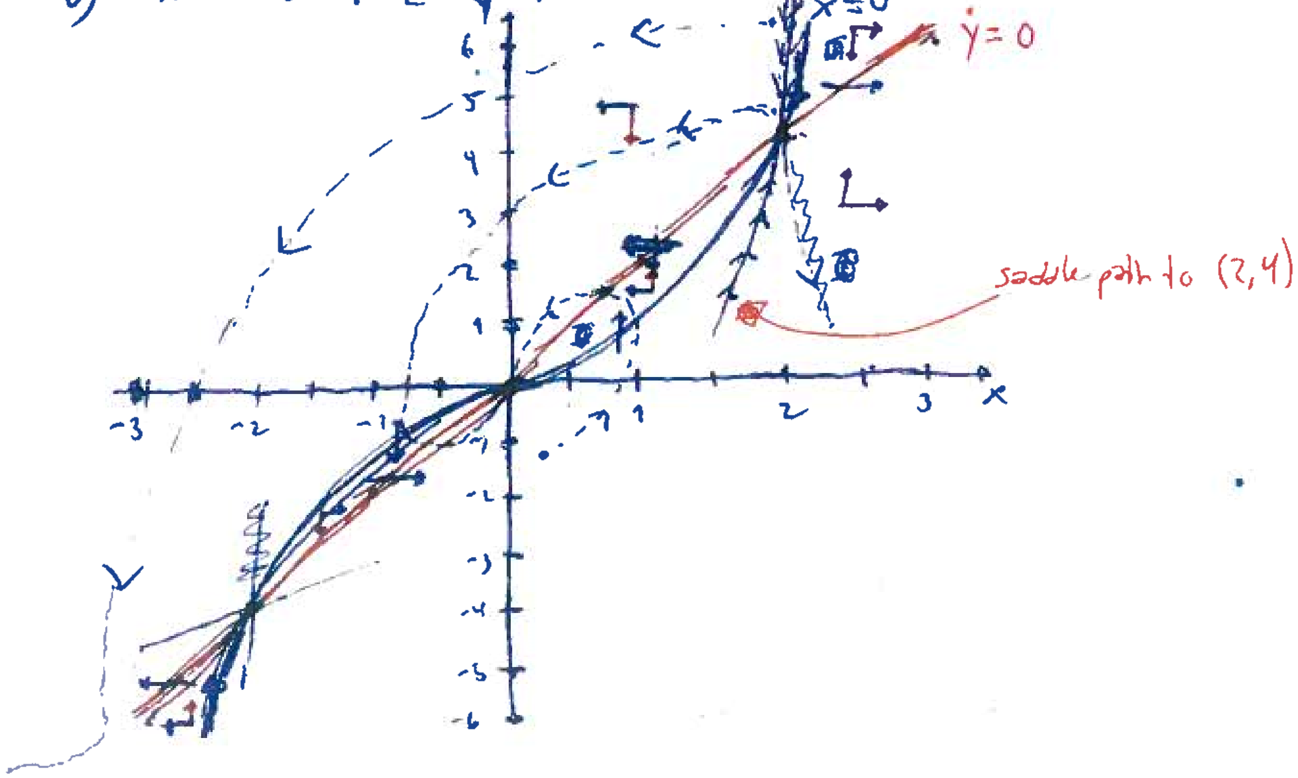
b)



b) $\dot{x}=0 \Rightarrow y=\frac{1}{2}x^3 \quad \dot{y}=0 \Rightarrow y=2x$



b) $\dot{x}=0 \Rightarrow y=\frac{1}{2}x^3 \quad \dot{y}=0 \Rightarrow y=2x$



c) This is a function $f(x)$ such that $f(2)=4$

and • $\frac{d}{dt}(f(x)) = \underline{f'(x)\dot{x} = 2x - f(x)}$ (1)

• $\underline{\dot{x} = \frac{1}{2}x^3 - f(x)}$ (2)

(2) in (1) gives $f'(x)\left(\frac{1}{2}x^3 - f(x)\right) = 2x - f(x) \Rightarrow \underline{f'(x) = \frac{2x - f(x)}{\frac{1}{2}x^3 - f(x)}}$

Thus $f'(2) = \frac{4-4}{0}$, use L'Hôpital's rule

gets : $f'(2) = \lim_{x\to 2} \frac{2 - f'(x)}{\frac{3}{2}x^2 - f(x)} = \frac{2 - f'(2)}{6 - f'(2)}$

This gives the equation $z(6-z) = 2-z$ i.e. $\underline{z^2 - 7z + 2 = 0}$

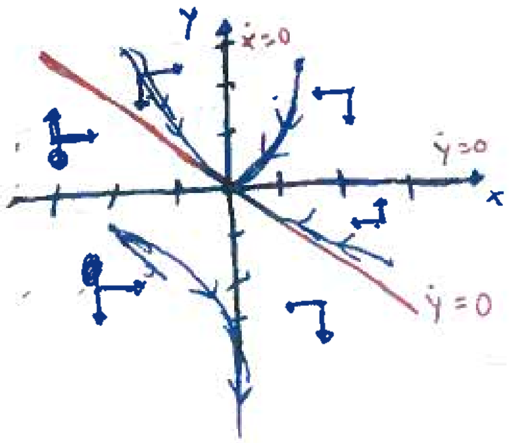
Solution: $z = \frac{7 \pm \sqrt{49-8}}{2} = \frac{7 \pm \sqrt{41}}{2}$ The slope will be positive (see drawing),

thus $\underline{\underline{f'(2) = \frac{7+\sqrt{41}}{2}}}$

Exercise 7.06

$\dot{x} = -x = f(x,y)$
$\dot{y} = -xy - y^2 = g(x,y)$

a) $\dot{x} = 0 \Rightarrow \underline{x=0}$ $\dot{y} = 0 \Rightarrow y^2 = -xy$ i.e. $\underline{y=0}$ or $\underline{y=-x}$



b) (0,0) only equilibrium point

$f_1' = -1 \quad f_2' = 0$

$g_1' = -y \quad g_2' = -x-2y$

$A = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow \mathrm{trace}(A) < 0$

$|A| = 0$

No result from this, but from the phase diagram one observes that (0,0) is not stable.

c) $\dot{x} = -x \quad \dot{x} + x = 0 \quad \frac{d}{dt}(xe^{t}) = 0 \quad xe^{t} = C \quad x = Ce^{-t}$

$x(0) = C = -1 \Rightarrow x = -e^{-t}$

$\Rightarrow \dot{y} = e^{-t}y - y^2 \quad \dot{y} - e^{-t}y = -y^2$ Bernoulli

$z = y^{-1} \quad -\dot{z} - e^{-t}z = -1 \quad \dot{z} + e^{-t}z = 1$

$\frac{d}{dt}(ze^{-e^{-t}}) = e^{-e^{-t}} \Rightarrow e^{-e^{-t}}z = \int e^{-e^{-s}}ds + C \quad z = e^{e^{-t}}\int_0^t e^{-e^{-s}}ds + e^{e^{-t}}C$

$\Rightarrow y = \frac{1}{e^{e^{-t}}\int_0^t e^{-e^{-s}}ds + e^{e^{-t}}C}$

$y(0) = \frac{1}{eC} = 1 \Rightarrow \boxed{C = \frac{1}{e}}$

$y = \frac{1}{e^{e^{-t}}\int_0^t e^{-e^{-s}}ds + e^{e^{-t}-1}}$

$\lim_{t\to\infty} x = 0 \quad \lim_{t\to\infty} y = 0$ (notice $e^{-e^{-s}} \to \infty$ as s grows).

↳ This means the point is a point on a saddle path.

~~ii) $x(0) = A + B = 0 \Rightarrow A = -B$; $y(0) = 2A - \frac{1}{2}B = -2B - \frac{1}{2}B = 0$; $-4B + B = 0$; $B = 0 \Rightarrow$~~

ii) We need $\lim_{t\to\infty} x = 0$ and $\lim_{t\to\infty} y = 0$

Only possible when $\boxed{A = 0}$

Thus $x^* = Be^{-t}$ $\quad y^* = -\frac{1}{2}Be^{-t}$ $\Rightarrow$ $\underline{\frac{x^*}{y^*} = -2}$ $\quad$ (Here, $B \neq 0$).

$\boxed{\text{Induction exercise}}$

a) $0 + 1 + 2 + \dots + n = \frac{n(n+1)}{2}$

• Base case $0 + 1 = 1$ $\quad \frac{1(1+1)}{2} = \frac{2}{2} = 1$ $\quad$ ok for base case

• Assume true for n, is then true for $n+1$?

$0 + 1 + \dots + n + (n+1) = \frac{n(n+1)}{2} + (n+1) = \frac{n(n+1)}{2} + \frac{2(n+1)}{2} = \frac{n(n+1) + 2(n+1)}{2} = \frac{(n+1)(n+2)}{2}$,

Yes, is true for $n+1$ as well $\Rightarrow$ <u>Hypothesis true</u>

b) • $\frac{9-1}{8} = \frac{8}{8} = 1$ $\Rightarrow$ Possible for base case.

• Assume true for n, is true for $n+1$? This means $\frac{9^n - 1}{8} = k$ ← integer

~~$\frac{9^{n+1}}{8} = \frac{9 \cdot 9^n}{8} - \frac{9}{8} - \frac{1}{8}$~~ or $9^n - 1 = 8k$ or $\underline{9^n = 8k + 1}$

Then, $\frac{9^{n+1} - 1}{8} = \frac{9 \cdot 9^n - 1}{8} = \frac{9 \cdot (8k+1) - 1}{8} = \frac{9 \cdot 8k + 9 - 1}{8}$

$= 9k + \frac{9}{8} - \frac{1}{8} = 9k + \frac{8}{8} = 9k + 1$ which is integer since k integer.

$\Rightarrow$ <u>Hypothesis true</u>