Exam 2011, Problem 3

a) $f(x_1,\dots,x_n) = (2\pi)^{-\frac{n}{2}} e^{\frac{-(x_1^2+\dots+x_n^2)}{2}}$

$e^{\cdot}$ increasing, and $-(x_1^2+\dots+x_n^2)$ concave $\Rightarrow$ is also quasiconcave

$\Rightarrow$ $\underline{\underline{f(x_1,\dots,x_n) \text{ quasiconcave}}}$

b) $\mathcal{L}(x,y) = \frac{1}{2\pi} e^{\frac{-(x^2+y^2)}{2}} - \lambda(-y) - \mu\left(y-(x-a)^{2011}\right)$

$$\boxed{\begin{array}{ll}
(1) & \frac{\partial \mathcal{L}}{\partial x} = -\frac{x}{2\pi} e^{\frac{-(x^2+y^2)}{2}} + 2011\mu(x-a)^{2010} = 0 \\
(2) & \frac{\partial \mathcal{L}}{\partial y} = -\frac{y}{2\pi} e^{\frac{-(x^2+y^2)}{2}} + \lambda - \mu = 0 \\
(3) & \lambda \geq 0 \quad (\lambda = 0 \text{ if } y > 0) \\
(4) & \mu \geq 0 \quad (\mu = 0 \text{ if } y < (x-a)^{2011}) \\
(5) & y \geq 0 \\
(6) & y \leq (x-a)^{2011}
\end{array}}$$

- Both binding $\underline{y=0}$ $y=(x-a)^{2011}$ $\Rightarrow$ $\underline{x=a}$

  (2) gives $\underline{\lambda=\mu}$ (1) gives $-\frac{a}{2\pi} e^{\frac{-a^2}{2}} = 0$ Impossible

- $\underline{y=0}$ $y<(x-a)^{2011}$ $\Rightarrow$ $\underline{x>a}$ $\Rightarrow$ $\underline{\mu=0}$

  (1) gives $x=0$ impossible (since $a>0$)

- $y>0$ $\underline{y=(x-a)^{2011}}$ $\Rightarrow$ $x>0$, $\underline{\lambda=0}$ (2) gives $\mu = -\frac{y}{2\pi} e^{\frac{-(x^2+y^2)}{2}} < 0$ Impossible

- None binding $y>0$ $y<(x-a)^{2011}$ $\Rightarrow$ $\underline{\lambda=0}$ $\underline{\mu=0}$ $\Rightarrow$ (1) gives $\underline{x=0}$

  (2) gives $y=0$ Impossible

$\underline{\underline{\text{No candidates}}}$

c) Optimizing f by choosing x and y as small as possible, that is choose both binding.

$\Rightarrow$ $(x^*,y^*) = (a,0)$. $\nabla g_1 = (0,-1)$ gradient of first constraint $\nabla g_2 = (-2011(x-a)^{2010}, 1)$ gradient of second constraint

With $(x^*,y^*) = (a,0)$ we get $\nabla g_1 = (0,-1)$ $\nabla g_2 = (0,1) = -\nabla g_1$

Since $\nabla g_2 = -\nabla g_1$, they are not linearly independent $\Rightarrow$ $\underline{\text{Constraint Qualification violated!}}$