Econ 4140/4145, 9/12-05

Answers to the problems

Problem 1

(a) Equilibrium points where $\dot{x} = \dot{y} = 0$. Note that $\dot{y} = 0$ iff $y^2 = \frac{1}{4}x^2$, i.e. $y = \pm\frac{1}{2}x$. Inserting $y = \frac{1}{2}x$ into $\dot{x} = 0$ yields $x^2 + 2x - 24 = 0$, with solutions $x = -6$ and $x = 4$. Thus $(-6, -3)$ and $(4, 2)$ are equilibrium points. Inserting $y = -\frac{1}{2}x$ into $\dot{x} = 0$ eventually yields $(6, -3)$ and $(-4, 2)$ as equilibrium points. So the 4 equilibrium points are: $(-6, -3)$, $(-4, 2)$, $(4, 2)$ and $(6, -3)$.

(b) With $f(x, y) = -\frac{1}{4}x^2 - y + 6$, $g(x, y) = -x^2 + 4y^2$, the Jacobian matrix is

$$J(x, y) = \begin{pmatrix} f_1'(x, y) & f_2'(x, y) \\ g_1'(x, y) & g_2'(x, y) \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}x & -1 \\ -2x & 8y \end{pmatrix}$$

Hence

$$J(4, 2) = \begin{pmatrix} -2 & -1 \\ -8 & 16 \end{pmatrix}$$

Since $|J(4, 2)| = -40 < 0$, $(4, 2)$ is a saddle point.

(c) A phase diagram looks roughly like this:

See the second figure on fig0305.pdf

Problem 2

(a) The Lagrangian is

$$\mathcal{L} = xy - \lambda_1(x^2 + ry^2) - \lambda_2(-x)$$

The necessary Kuhn–Tucker conditions for (x^*, y^*) to solve the problem are:

$\mathcal{L}_1' = y^* - 2\lambda_1 x^* + \lambda_2 = 0$

$\mathcal{L}_2' = x^* - 2r\lambda_1 y^* = 0$

$\lambda_1 \geq 0 \quad (\lambda_1 = 0 \quad \text{if} \quad (x^*)^2 + r(y^*)^2 < m)$

$\lambda_2 \geq 0 \quad (\lambda_2 = 0 \quad \text{if} \quad x^* > 1)$

$(x^*)^2 + r(y^*)^2 \leq m$

$x^* \geq 1$

(b) From (2) and (6) we see that $\lambda_1 = 0$ is impossible. Thus $\lambda_1 > 0$ and from (3) and (5),

$$(x^*)^2 + r(y^*)^2 = m \tag{7}$$

(Forts.)

Case I, $\lambda_2 = 0$. Then from (1) and (2), $y^* = 2\lambda_1 x^*$ and $x^* = 2\lambda_1 r y^*$, so $y^* = 4\lambda_1^2 r y^*$. If $y^* = 0$, then (2) implies $x^* = 0$, which is impossible. Hence, $\lambda_1^2 = 1/4r$ and thus $\lambda_1 = 1/2\sqrt{r}$. Then $y^* = x^*/\sqrt{r}$, which inserted into (7) and solved for x^* yields $x^* = \sqrt{m/2}$ and then $y^* = \sqrt{m/2r}$. Note that $x^* \geq 1$ iff $\sqrt{m/2} \geq 1$ iff $m \geq 2$.

Thus for $m \geq 2$, $x^* = \sqrt{m/2}$ and $y^* = \sqrt{m/2r}$, with $\lambda_1 = 1/2\sqrt{r}$ and $\lambda_2 = 0$ is a solution candidate.

Case II, $\lambda_2 > 0$. Then $x^* = 1$ and from (7) we have $r(y^*)^2 = m - 1$, so $y^* = \sqrt{(m-1)/r}$ ($y^* = -\sqrt{(m-1)/r}$ contradicts (2)). Inserting these values for x^* and y^* into (1) and (2) and solving for λ_1 and λ_2 yields $\lambda_1 = 1/2\sqrt{r(m-1)}$ and furthermore, $\lambda_2 = (2-m)/\sqrt{r(m-1)}$. Note that $\lambda_2 > 0$ iff $m < 2$.

Thus, for $1 < m < 2$, the only solution candidate is $x^* = 1$, $y^* = \sqrt{(m-1)/r}$, with $\lambda_1 = 1/2\sqrt{r(m-1)}$ and $\lambda_2 = (2-m)/\sqrt{r(m-1)}$.

The objective function is continuous and the constraint set is obviously closed and bounded, so by the extreme value theorem there has to be a maximum. The solution candidates we have found are therefore optimal. (Alternatively, look at the Hessian matrix $\mathbf{H}(x,y) = \begin{pmatrix} \mathcal{L}''_{11}(x,y) & \mathcal{L}''_{12}(x,y) \\ \mathcal{L}''_{21}(x,y) & \mathcal{L}''_{22}(x,y) \end{pmatrix} = \begin{pmatrix} -2\lambda_1 & 1 \\ 1 & -2r\lambda_1 \end{pmatrix}$. Here $\mathcal{L}''_{11}(x,y) = -2\lambda_1 \leq 0$, $\mathcal{L}''_{22}(x,y) = -2r\lambda_1 \leq 0$, and $|\mathbf{H}(x,y)| = 4r\lambda_1^2 - 1$. In the case $m \geq 2$, $|\mathbf{H}(x,y)| = 0$, and in the case $1 < m < 2$, $|\mathbf{H}(x,y)| = (2-m)/(m-1) > 0$. Thus in both cases, $\mathcal{L}(x,y)$ is concave.)

Problem 3

With $F(t,x,\dot{x}) = (tx - x^2 - t^2\dot{x} - \frac{1}{2}\dot{x}^2)e^{-t}$,

$$\frac{\partial F}{\partial x} = (t-2x)e^{-t}, \qquad \frac{\partial F}{\partial \dot{x}} = (-t^2 - \dot{x})e^{-t}$$

and the Euler equation $\partial F/\partial x - \frac{d}{dt}(\partial F/\partial \dot{x}) = 0$ becomes

$$(t-2x)e^{-t} - \frac{d}{dt}(-t^2 - \dot{x})e^{-t} = 0$$

or

$$(t-2x)e^{-t} + (2t + \ddot{x})e^{-t} - (t^2 + \dot{x})e^{-t} = 0$$

Cancelling e^{-t} and rearranging, we obtain the Euler equation

$$\ddot{x} - \dot{x} - 2x = t^2 - 3t$$

(Forts.)

Problem 4

(a) With the Hamiltonian $H(t, x, u, p) = x - e^{-u} - pu$, the necessary conditions for $(x^*(t), u^*(t))$ to solve the problem are that there exists a continuous function $p = p(t)$ such that:

(I) $u^*(t)$ maximizes $x^*(t) - e^{-u} - p(t)u$ for $u \leq 1$

(II) $\dot{p}(t) = -\partial H^*/\partial x = -1$, and $p(1) = 0$

(III) $\dot{x}^*(t) = -u^*(t)$, $x^*(0) = 0$

The Hamiltonian is concave as a sum of concave functions, so conditions (I)–(III) are sufficient for optimality.

(b) From (II), $p(t) = 1 - t$, and according to (I), for each $t \in [0, 1]$, $u^*(t)$ must maximize $h(u) = (t - 1)u - e^{-u}$ for $u \leq 1$. The function $h(u)$ is concave, so the maximum must be at $u = 1$ if $h'(1) = t - 1 + e^{-1} \geq 0$. This occurs if $t \geq 1 - 1/e$. For $t < 1 - 1/e$, maximum occurs when $h'(u) = 0$, i.e. for $u = -\ln(1 - t)$. Our suggestion for an optimal control is therefore

$$u^*(t) = \begin{cases} -\ln(1-t) & \text{if } t \in [0, 1 - 1/e] \\ 1 & \text{if } t \in (1 - 1/e, 1] \end{cases}$$

In the interval $[0, 1-1/e]$, $\dot{x}^*(t) = \ln(1-t)$, so $x^*(t) = \int(\ln(1-t)\,dt$. Introduce $z = 1 - u$ as a new variable. Then $x^*(t) = \int \ln(1-t)\,dt = -\int \ln z\,dz = -z\ln z + z + C = -(1-t)\ln(1-t) + (1-t) + C$. Since $x^*(0) = 0$, we get $\mathrm{x}^*(t) = -(1-t)\ln(1-t) - t$. In particular, $x^*(1 - 1/e) = 2/e - 1$.

In the interval $(1 - 1/e, 1]$, $\dot{x}^*(t) = -1$, so with $x^*(1 - 1/e) = 2/e - 1$, so $x^*(t) = -t + 1/e$. Thus

$$x^*(t) = \begin{cases} -(1-t)\ln(1-t) - t & \text{if } t \in [0, 1 - 1/e] \\ -t + 1/e & \text{if } t \in (1 - 1/e, 1] \end{cases}$$

Since H is concave in (x, u), we have found the solution.