University of Oslo
Department of Economics

ECON4140/4145 Mathematics 3

Monday 10 December 2007, 14:30–17:30.

There are 2 pages of problems to be solved.

All printed and written material may be used. Pocket calculators are allowed.

State reasons for all your answers.

Grades given: A (best), B, C, D, E, F, with E as the weakest passing grade.

Problem 1

Consider the following system of differential equations:

$$\text{(S)} \qquad \begin{aligned} \dot{x} &= 2x - x^2 - y + 2 \\ \dot{y} &= x - y \end{aligned}$$

(a) Find all equilibrium points for the system (S) and determine for each of them whether it is locally asymptotically stable, a saddle point, or neither.

(b) Draw a phase diagram for the system in the first quadrant (i.e. for $x > 0$, $y > 0$), and try to sketch some possible solution curves.

Problem 2

Consider the matrix $\mathbf{A} = \begin{pmatrix} -1 & -1 & 1/3 \\ 2 & 2 & -2/3 \\ -2 & -1 & 2/3 \end{pmatrix}$.

(a) Show that the vectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$, and $\mathbf{v}_3 = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$ are eigenvectors of $\mathbf{A}$.

(b) Let $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ be the eigenvectors given in part (a). Show that the vector $\mathbf{w} = \begin{pmatrix} 4 \\ 0 \\ 5 \end{pmatrix}$ can be written as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and use this to describe what happens to $\mathbf{A}^n\mathbf{w}$ as $n \to \infty$.

(Cont.)

Problem 3

Consider the following nonlinear programming problem:

$$\text{maximize } -(x-6)^2-(y-5)^2 \quad \text{subject to} \quad \begin{cases} x^2+y^2 \le 25 \\ a(x-3)+y \le 4 \end{cases}$$

where a is a parameter, $a \neq 3/4$.

(a) Write out the Kuhn–Tucker conditions for a point (x, y) to be the optimal solution of the problem.

(b) For what values of the parameter a will $(x, y) = (3, 4)$ be the optimal solution?
(*Hint:* Look at the Lagrange multipliers in the Kuhn–Tucker conditions. You can take for granted that the constraint qualification (Norwegian: "føringsbetingelsen") is satisfied.)

Problem 4

Consider the following optimal control problem:

$$\max \int_0^T (ux - u^2 - \tfrac{1}{4}x^2)e^{-t}\,dt, \quad \dot{x} = 2u, \;\; x(0) = x_0, \;\; x(T) = x_T, \;\; u \in (-\infty, \infty).$$

(a) Write out the necessary conditions from the maximum principle for an admissible pair (x^*, u^*) to be optimal. (You can take for granted that $p_0 = 1$.)

(b) Solve the problem.