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Solutions of the examination problems in ECON4140/4145 Mathematics 3, 10 December 2007

Problem 1

(a) Let $f(x, y) = 2x - x^2 - y + 2$ and $g(x, y) = x - y$. The equilibrium points (x, y) are the solutions of the equation system

$$\begin{aligned} 2x - x^2 - y + 2 &= 0 \\ x - y &= 0 \end{aligned}$$

The second equation yields $y = x$, and then the first equation implies $x - x^2 + 2 = 0$, which has the roots $x_1 = -1$, $x_2 = 2$. Thus the equilibrium points are $(x_1, y_1) = (-1, -1)$ and $(x_2, y_2) = (2, 2)$.

The Jacobian matrix of f and g with respect to x and y is

$$\mathbf{A}(x, y) = \begin{pmatrix} 2 - 2x & -1 \\ 1 & -1 \end{pmatrix}.$$

At the equilibrium points $\mathbf{A}(x, y)$ becomes

$$\mathbf{A}_1 = \mathbf{A}(-1, -1) = \begin{pmatrix} 4 & -1 \\ 1 & -1 \end{pmatrix}, \qquad \mathbf{A}_2 = \mathbf{A}(2, 2) = \begin{pmatrix} -2 & -1 \\ 1 & -1 \end{pmatrix}.$$

We have $\det(\mathbf{A}_1) = -3 < 0$, so $(-1, -1)$ is a saddle point. The equilibrium point $(2, 2)$ is locally asymptotically stable, because $\det(\mathbf{A}_2) = 3 > 0$ and $\operatorname{tr}(\mathbf{A}_2) = -3 < 0$.

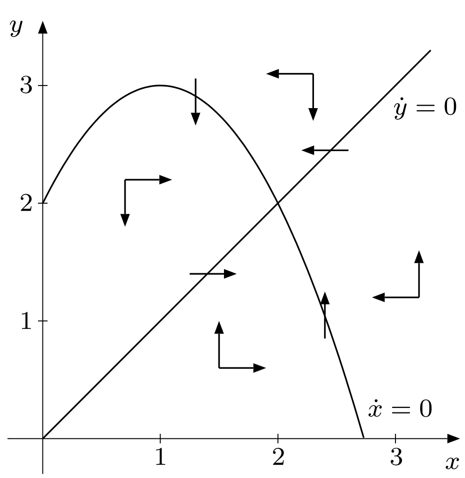

Figure 1

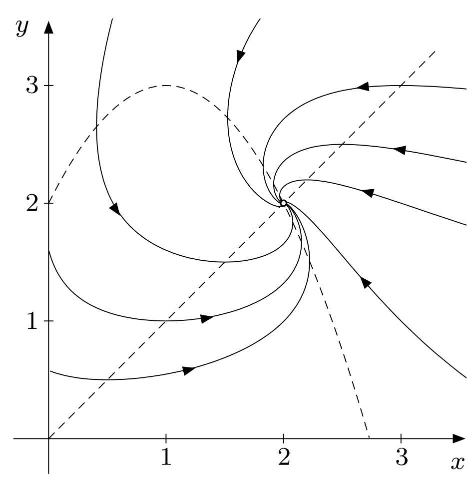

Figure 2

(b) The nullclines $\dot{x} = 0$ and $\dot{y} = 0$ are given by the equations $y = 2x - x^2 + 2$ and $y = x$, respectively. Figure 1 above shows the nullclines and arrows indicating the signs of $\dot{x}$ and $\dot{y}$ in each of the four regions into which the nullclines divide the first quadrant of the xy-plane. Figure 2 shows some of the solution paths.

Problem 2

(a) Straightforward matrix multiplication shows that

$$\mathbf{A}\mathbf{v}_1 = \mathbf{0} = 0\mathbf{v}_1\,, \quad \mathbf{A}\mathbf{v}_2 = \tfrac{2}{3}\mathbf{v}_2\,, \quad \mathbf{A}\mathbf{v}_3 = \mathbf{v}_3 = 1\mathbf{v}_3\,.$$

It follows that all the $\mathbf{v}$'s are eigenvectors of $\mathbf{A}$, corresponding to the eigenvalues

$$\lambda_1 = 0\,, \quad \lambda_2 = \tfrac{2}{3}\,, \quad \lambda_3 = 1\,.$$

(b) We want to find coefficients c_1, c_2, c_3 such that $\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$. This vector equation is equivalent to the system

$$\begin{aligned} c_1 + c_2 + c_3 &= 4 \\ -2c_2 - 2c_3 &= 0 \\ 3c_1 - c_2 &= 5 \end{aligned}$$

The system is easily solved by "unsystematic elimination": The second equation yields $c_3 = -c_2$, and the first equation gives $c_1 = 4$. The last equation tells us that $c_2 = 3c_1 - 5 = 7$, and then $c_3 = -c_2 = -7$. It follows that

$$\mathbf{w} = 4\mathbf{v}_1 + 7\mathbf{v}_2 - 7\mathbf{v}_3\,.$$

Since the $\mathbf{v}$'s are eigenvectors, we have

$$\mathbf{A}\mathbf{w} = \mathbf{A}(4\mathbf{v}_1 + 7\mathbf{v}_2 - 7\mathbf{v}_3) = 4\lambda_1\mathbf{v}_1 + 7\lambda_2\mathbf{v}_2 - 7\lambda_3\mathbf{v}_3 = 7(\tfrac{2}{3})\mathbf{v}_2 - 7\mathbf{v}_3\,,$$

and, for $n \geq 1$,

$$\mathbf{A}^n\mathbf{w} = 4\lambda_1^n + 7\lambda_2^n\mathbf{v}_2 - 7\lambda_3^n\mathbf{v}_3 = 7(\tfrac{2}{3})^n\mathbf{v}_2 - 7\mathbf{v}_3\,.$$

Since $(\frac{2}{3})^n \to 0$ as $n \to \infty$, we have

$$\lim_{n\to\infty} \mathbf{A}^n\mathbf{w} = -7\mathbf{v}_3\,.$$

Problem 3

(a) With the Lagrangian

$$\mathcal{L}(x,y) = -(x-6)^2 - (y-5)^2 - \lambda(x^2 + y^2 - 25) - \mu(a(x-3) + y - 4)\,,$$

the Kuhn–Tucker conditions are

$$\begin{aligned} \mathcal{L}_1'(x,y) &= -2x + 12 - 2\lambda x - a\mu = 0 \\ \mathcal{L}_2'(x,y) &= -2y + 10 - 2\lambda y - \mu = 0 \\ \lambda \geq 0, &\quad \lambda = 0 \text{ if } x^2 + y^2 < 25 \\ \mu \geq 0, &\quad \mu = 0 \text{ if } a(x-3) + y < 4 \end{aligned}$$

(b) The Lagrangian is concave, so a point that satisfies the Kuhn–Tucker conditions will certainly be optimal. At the point $(3,4)$ both constraints are active, and the Kuhn–Tucker conditions reduce to

$$6 - 6\lambda - a\mu = 0 \qquad (1)$$

$$2 - 8\lambda - \mu = 0 \qquad (2)$$

$$\lambda \geq 0, \quad \mu \geq 0 \qquad (3)$$

The point $(3,4)$ will be optimal if and only if the equations (1) and (2) give nonnegative values for λ and μ.

We rewrite (1) and (2) as

$$6\lambda + a\mu = 6 \qquad (4)$$

$$8\lambda + \mu = 2 \qquad (5)$$

The determinant of this system is $6 - 8a$, which is different from 0 as long as $a \neq 3/4$. Therefore the system has a unique solution, which turns out to be

$$\lambda = \frac{a-3}{4a-3}, \qquad \mu = \frac{18}{4a-3}.$$

It is now clear that

$$(\lambda \geq 0 \ \& \ \mu \geq 0) \iff a \geq 3.$$

Hence, the point $(3,4)$ is optimal if and only if $a \geq 3$.

Comment: The constraints are both active at $(3,4)$, and the corresponding gradients there are $(6,8)$ and $(a,1)$. These gradients are linearly independent if and only if $a \neq 3/4$, and therefore the constraint qualification holds at $(3,4)$ whenever $a \neq 3/4$. If $a = 3/4$, the constraint qualification fails. The geometric reason is that the straight line $a(x-3)+4$ then becomes tangent to the circle $x^2 + y^2 = 25$ at $(3,4)$. It is easy to see, however, that $(3,4)$ is not optimal in this case: The admissible set will be the entire circular disk determined by the first constraint, and the optimal point is the point in the disk that is closest to $(6,5)$, namely the point $(30/\sqrt{61},\ 25/\sqrt{61})$.)

Problem 4

(a) With the Hamiltonian

$$H(t,x,u,p) = (ux - u^2 - \tfrac{1}{4}x^2)e^{-t} + 2pu$$

the maximum principle yields the following necessary conditions for an admissible pair (x^*, u^*) to be optimal: There must exist a continuous and piecewise differentiable function $p = p(t)$ such that

(1) For each t in $[0,T]$, $u = u^*(t)$ must maximize $H(t, x^*(t), u, p(t))$ for u in $\mathbb{R}$.

(2) $\dot{p}(t) = -H'_x(t, x^*(t), u^*(t), p(t))$ for all t in $[0,T]$, except possibly where u^* is discontinuous.

There is no transversality condition, because $x(T)$ is fixed.

In the present problem $H(t,x,u,p)$ is differentiable and concave with respect to u, and u can take all real values. Therefore the maximum is attained at a point where $H'_u = 0$. Since

$$H'_u(t,x,u,p) = (x-2u)e^{-t} + 2p\,,$$

condition (1) reduces to

$$\text{(1a)} \qquad u^*(t) = \tfrac{1}{2}x^*(t) + e^t p(t)\,.$$

Then (2) implies

$$\text{(2a)} \qquad \dot{p}(t) = -\big[u^*(t) - \tfrac{1}{2}x^*(t)\big]e^{-t} = -p(t)\,.$$

(b) In general, the maximum principle only gives necessary conditions. But in our case, the Hamiltonian can be written as

$$H(t,x,u,p) = -(u-\tfrac{1}{2}x)^2 e^{-t} + 2pu\,,$$

which is obviously *concave* w.r.t (x,u), and therefore any admissible pair (x^*,u^*) that satisfies the conditions in the maximum principle will yield an optimal solution. So let us see what we can find.

From (2a) we get

$$\text{(3)} \qquad p(t) = Ce^{-t}\,.$$

It then follows from (1a) that $u^*(t) - \frac{1}{2}x^*(t) = p(t)e^t = C$, so $u^*(t) = C + \frac{1}{2}x^*(t)$. Therefore

$$\dot{x}^*(t) = 2u^*(t) = 2C + x^*(t) \iff \dot{x}^*(t) - x^*(t) = 2C,$$

with the general solution

$$\text{(4)} \qquad x^*(t) = Ae^t - 2C\,.$$

Hence

$$\text{(5)} \qquad u^*(t) = \tfrac{1}{2}\dot{x}^*(t) = \tfrac{1}{2}Ae^t\,.$$

We now know u^*, x^* and p, and it only remains to find the constants A and C. The boundary conditions $x(0) = x_0$ and $x(T) = x_T$ yield the equations $A - 2C = x_0$ and $Ae^T - 2C = x_T$. These equations are easily solved to give

$$A = \frac{x_T - x_0}{e^T - 1}\,, \quad C = \frac{x_T - x_0 e^T}{2(e^T - 1)}\,.$$

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