University of Oslo / Department of Economics **(English only)**

# ECON4140 / ECON 4145 Mathematics 3

Monday December 8 2008, 09:00–12:00

There are 2 pages of problems to be solved.

All printed and written material may be used, as well as pocket calculators.

Give reasons for all your answers.

Grades given run from A (best) to E for passes, and F for fail.

## Problem 1

(a) Consider for each $k \in \mathbf{R}$ the matrices

$$\mathbf{C}_k = \begin{pmatrix} k & 2 & k \\ 2 & 3 & 0 \\ k & 0 & k \end{pmatrix} \quad \text{and} \quad \mathbf{D}_k = \begin{pmatrix} k & 2 & k & k \\ 2 & 3 & 0 & k+1 \\ k & 0 & k & ke^k \end{pmatrix} = \begin{pmatrix} & \vdots & k \\ \mathbf{C}_k & \vdots & k+1 \\ & \vdots & ke^k \end{pmatrix}$$

Find the rank of $\mathbf{C}_k$ and the rank of $\mathbf{D}_k$, and decide for what value(s) of $k$ the equation system $\mathbf{C}_k \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} k \\ k+1 \\ ke^k \end{pmatrix}$ has a solution.

(b) For the rest of problem 1, let $\mathbf{A} = \mathbf{C}_0 = \begin{pmatrix} 0 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

i) $\lambda_1 = 4$ is an eigenvalue for $\mathbf{A}$. Find a corresponding eigenvector.

ii) $\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$ is an eigenvector for $\mathbf{A}$. Find the corresponding eigenvalue $\lambda_2$.

iii) Find the third eigenvalue $\lambda_3$ (so that $\lambda_1 \neq \lambda_3 \neq \lambda_2$) and a corresponding eigenvector.

(c) Let $\mathbf{A}$ be given as in part (b) above.

i) Show that the quadratic form $\begin{pmatrix} x & y & z \end{pmatrix} \mathbf{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is indefinite.

ii) Decide the definiteness of $\begin{pmatrix} x & y & z \end{pmatrix} \mathbf{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ subject to $\begin{pmatrix} 16 & 0 & 1 \\ 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{0}$.

(Problem 1 continued next page)

(d) Let $\mathbf{A}$ be given as in part (b) above. Consider the differential equation system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}.$$

i) Solve the system.
(*Hint:* If you do not wish to use the results from part (b) above, then solve first for $x_3(t)$ and then for $\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$.)

ii) The origin is a saddle point for the system. Let $\begin{pmatrix} x_1^*(t) \\ x_2^*(t) \\ 0 \end{pmatrix}$ be a solution curve which converges to the origin. Find $\frac{x_1^*(t)}{x_2^*(t)}$ provided that $x_2^*(t) \neq 0$.

**Problem 2** Consider the dynamic programming problem

$$\max \sum_{t=0}^{T} (e^{x_t} + u_t), \quad \text{where } x_{t+1} = x_t - \ln u_t, \qquad u_t \in [1, e]$$

and where $x_0 \geq T + 1$ (so that $x_t \geq 1$ always).

Let $J_t(x)$ be the value function, i.e. the maximal $\sum_{s=t}^{T} (e^{x_s} + u_s)$.

(a) Calculate $J_{T-1}(x)$, $J_{T-2}(x)$ and $J_{T-3}(x)$.
(*Hint:* Show that $u_{T-1}^* = u_{T-2}^* = u_{T-3}^* = 1$ is optimal. If you do not manage to show this, then use it nevertheless for partial score.)

(b) Show that for each $t = 0, 1, \ldots, T-1, T$, the value function is of the form $J_t(x) = A_t e^x + B_t$, where $A_t \geq 1$.

**Problem 3** Consider the optimal control problem

$$\max \int_0^{30} u x^2 \, dt, \qquad \text{where } \dot{x} = u \in [-5, -2], \qquad x(0) = 60, \qquad x(30) \geq -90$$

(a) State the necessary conditions from the maximum principle.

(b) Show that for the optimal $x^*(t)$ we have

$$(x^*(t))^2 = -p(t),$$

where $p$ is the adjoint (costate) from the maximum principle.
(*Hint:* Put $\alpha(t) = p(t) + p_0 (x^*(t))^2$ and calculate $\dot{\alpha}$. Then show that $\alpha > 0$ is impossible, and that $\alpha < 0$ is impossible. With $\alpha = 0$, can then $p_0 = 0$?)

(c) Find an approximate change in optimal value obtained by replacing the initial condition by $x(0) = 61$. You can take for granted that the value function is differentiable.
(*Hint:* You are not supposed to solve the optimal control problem.)