ECON 4140 / ECON 4145 fall 2008: The exam solved

Problem 1

(a) $|\mathbf{C}_k| = k \cdot (-3k + 3k - 4) = -4k$, so that for $k \neq 0$, both $\mathbf{C}_k$ and $\mathbf{D}_k$ have rank three. For $k = 0$ the upper-left 2×2 minor $\left|\begin{smallmatrix} 0 & 2 \\ 2 & 3 \end{smallmatrix}\right|$ is nonzero, while – by the zero row – any 3×3 minor is zero. So the ranks are equal for each k, and

> For $k \neq 0$, both $\mathbf{C}_k$ and $\mathbf{D}_k$ have rank 3.
> For $k = 0$, both $\mathbf{C}_k$ and $\mathbf{D}_k$ have rank 2.
> The equation system always has a solution.

(b) i) $\lambda_1 = 4$ is an eigenvalue for $\mathbf{A}$. A corresponding eigenvector is one that solves

$$\begin{pmatrix} -4 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \mathbf{0}$$

$v_3 = 0$ and the two first rows are proportional; putting $v_1 = \theta_1$, we get $v_2 = 2\theta_1$ so that an eigenvector will be e.g.

$$\boxed{\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}}$$

(or any nonzero scalar multiple.)

ii) Direct calculation yields $\mathbf{A}(2, -1, 0)' = (-2, 1, 0)' = (-1)(2, -1, 0)'$ so that the vector is an eigenvector with corresponding eigenvalue

$$\boxed{\lambda_2 = -1.}$$

iii) From part (a), we know that $|\mathbf{A}| = 0$, so zero is an eigenvalue. A corresponding eigenvector is one which solves $\mathbf{A}(w_1, w_2, w_3)' = \mathbf{0}$, from which we see that $w_1 = 0$, hence $w_2 = 0$ while w_3 is free. So the eigenvalue is

$$\boxed{\lambda_3 = 0 \text{ with corresponding eigenvector } \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}$$

(where any scalar multiple of $(0, 0, 1)'$ is also an eigenvector.)

(c) i) Using the eigenvalues from part (b) above, we have $\lambda_1 > 0 > \lambda_2$, so the quadratic form is indefinite, Q.E.D.

ii) Consider the bordered matrix $\mathbf{M} = \begin{pmatrix} \multicolumn{2}{c}{\mathbf{0}} & 16 & 0 & 1 \\ & & 1 & 2 & 0 \\ 16 & 1 & & & \\ 0 & 2 & & \mathbf{A} & \\ 1 & 0 & & & \end{pmatrix}$. With two constraints (composed of linearly independent vectors, but this linear dependence will follow from the calculations), we only need consider minors of order > 4, i.e. only $|\mathbf{M}|$ itself. Cofactor expansion along the last row gives

$$|\mathbf{M}| = \begin{vmatrix} 0 & 16 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 1 & 0 & 2 & 0 \\ 2 & 2 & 3 & 0 \end{vmatrix} = -\begin{vmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ 2 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} - 2\begin{vmatrix} 1 & 0 \\ 2 & 2 \end{vmatrix} = -1 - 4 < 0.$$

[Bug fixed]

So $(-1)^3|\mathbf{M}| > 0$, and

> the quadratic form is *negative definite* subject to the constraint.

(d) i) From part (b), using the eigenvalues and eigenvectors, we get that the solution can be represented as

$$\boxed{\mathbf{x}(t) = C_1 e^{4t}\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + C_2 e^{-t}\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + C_3\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}$$

for arbitrary constants C_1, C_2 and C_3.

ii) A solution curve converging to the origin must have $C_1 = C_3 = 0$. Then

$$\frac{x_1^*(t)}{x_2^*(t)} = \frac{2C_2 e^{-t}}{-C_2 e^{-t}} = \boxed{-2}$$

provided that $x_2^*(t) \neq 0$.

Problem 2

(a) We have $J_T(x) = \max_{u\in[1,e]}\{e^x + u\} = e^x + e$. Then

$$\begin{aligned} J_{T-1}(x) &= \max_{u\in[1,e]}\{e^x + u + J_T(x - \ln u)\} \\ &= \max_{u\in[1,e]}\{e^x + u + e^{x-\ln u} + e\} \\ &= e^x + e + \max_{u\in[1,e]} \underbrace{\{u + \frac{e^x}{u}\}}_{\text{convex in } u} \end{aligned}$$

so that the optimal u^* is either 1 or e. 1 is optimal if and only if

$$1 + \frac{e^x}{1} \geq e + \frac{e^x}{e} \qquad \text{which } \Leftrightarrow \qquad e^x \geq \frac{e-1}{1-e^{-1}} = e$$

which is true for all $x \geq 1$ (and J_{T-1} is undefined for $x < 1$). So

$$\begin{aligned} J_{T-1}(x) &= e^x + e + 1 + e^x \\ &= \boxed{2e^x + 1 + e} \end{aligned}$$

Proceeding analogously, we have

$$J_{T-2}(x) = e^x + 1 + e + \max_{u\in[1,e]} \underbrace{\{u + \frac{2e^x}{u}\}}_{\text{convex in } u}$$

and as above, $u^* = 1$ is optimal if and only if

$$1 + \frac{2e^x}{1} \geq e + \frac{2e^x}{e} \qquad \text{which } \Leftrightarrow \qquad 2e^x \geq e$$

which is true, so

$$\begin{aligned} J_{T-2}(x) &= e^x + 1 + e + 2e^x + 1 \\ &= \boxed{3e^x + 2 + e} \end{aligned}$$

Analogously we have

$$J_{T-3}(x) = e^x + 2 + e + \max_{u\in[1,e]} \underbrace{\{u + \frac{3e^x}{u}\}}_{\text{convex in } u}$$

and as above, $u^* = 1$ is optimal because for all $x \geq 1$ we have $1 + \frac{2e^x}{1} \geq e$, so that

$$\begin{aligned} J_{T-3}(x) &= e^x + 2 + e + 3e^x + 1 \\ &= \boxed{4e^x + 3 + e} \end{aligned}$$

(b) For $t = T$, we have $J_T(x) = e^x + \max_{u\in[1,e]} u = e^x + e$, which satisfies the requested form (with $A_T = 1$ and $B_T = u_T^* = e$). So assume the given form applies for J_t, where $t \in \{1, \dots, T\}$ is arbitrary. Then for $t-1$ we have

$$\begin{aligned} J_{t-1}(x) &= e^x + \max_{u\in[0,1]} \{u + J_t(x - \ln u)\} \\ &= e^x + \max_{u\in[0,1]} \{u + A_t e^{x-\ln u} + B_t\} \\ &= e^x + B_t + \max_{u\in[0,1]} \{u + \frac{1}{u} A_t e^x\}. \end{aligned}$$

By assumption, $A_t \geq 1$ so that the braced expression is convex in u, and hence maximized by an endpoint. $u = 1$ gives $1 + A_t e^x$ while $u = e$ gives $e + \frac{1}{e} A_t e^x$. The former is maximal if

$$e + eA_t e^x \geq e^2 + A_t e^x \qquad \Leftrightarrow \qquad (e-1)A_t e^x \geq (e-1)e \qquad \Leftrightarrow \qquad A_t e^x \geq e$$

which is always true, since $x \geq 1$ (given in the problem) and $A_t \geq 1$ by the induction hypothesis. So for all $t < T$, it is optimal to choose $u_t^* = 1$, and we obtain

$$J_{t-1}(x) = e^x + B_t + 1 + A_t e^x = (A_t + 1)e^x + B_t + 1.$$

so that $A_{t-1} = A_t + 1 \geq 1$. The proof is complete.

Problem 3

(a) With $H(t, x, u, p) = p_0 u x^2 + pu$, the maximum principle gives the conditions that

> u^* maximizes $u \cdot (p_0 x^2 + p)$ $\left(\text{so that } u^* = \begin{cases} -2 & \text{if } p_0(x^*)^2 + p > 0 \\ -5 & \text{if } p_0(x^*)^2 + p < 0 \end{cases}\right)$
>
> p satisfies $\dot{p} = -2p_0 u^* x^*$, with the transversality condition
> $p(30) \geq 0$ (and $p(30) = 0$ if $x(30) > 90$)
> $p_0 = 0$ or $p_0 = 1$ (and $p_0 = 1$ if $p(t) = 0$ for some t).

(b) From the hint, we get $\dot{\alpha} = \dot{p} + 2p_0 x\dot{x} = 0$. So α is a constant A. Evaluating at T yields $A = \alpha(T) = p(T) + p_0(x^*(T))^2$ which is ≥ 0 by the transversality condition, so $A \geq 0$. Assume for contradiction that $A > 0$. Then $u^* = -2$ always, so that $x(T) = x(0) - 60 = 0$ which is > -90 so that $p(T) = 0$ implying $A = 0$.
So $p_0 x^*(t)^2 + p(t) = 0$ for all t. We cannot have $p_0 = 0$, because that implies $p = 0$ too. So $p_0 = 1$ and we have $x^*(t)^2 = -p(t)$, Q.E.D.

(c) A unit increase in initial state is approx. equal to the derivative of the value function with respect to initial state, which equals $p(0)$, which by part (b) is

$$p(0) = -(x^*(0))^2 = -60^2 = \boxed{-3600}$$