University of Oslo
Department of Economics

ECON4140/4145 Mathematics 3

Wednesday 9 December 2009, 9:00–12:00.

There are 2 pages of problems to be solved.

All printed and written material may be used. Pocket calculators are allowed.

State reasons for all your answers.

Grades given: A (best), B, C, D, E, F, with E as the weakest passing grade.

Problem 1

Consider the quadratic form

$$Q(x, y, z) = x^2 + y^2 + z^2 + 2axy + 2xz + 2yz\,.$$

(a) Prove that Q is not positive definite for any value of a.

(b) Prove that Q is positive semidefinite for one particular value of a.

(c) Put $a = 1$ and find the eigenvalues of the symmetric matrix associated with Q.

Problem 2

Consider the following system of differential equations:

$$\begin{aligned} \dot{x} &= y - x \\ \dot{y} &= x + \tfrac{1}{8}x^3 + \tfrac{1}{2}y \end{aligned}$$

(a) Show that the system has exactly one equilibrium point and that the equilibrium point is a saddle point.

(b) Find an eigenvector corresponding to the negative eigenvalue of the Jacobian matrix at the equilibrium point.

(c) Draw a phase diagram and sketch some possible solution curves. Indicate in the diagram the solution curves that converge to the equilibrium point.

(Cont.)

Problem 3

Consider the variational problem

$$\max \int_0^1 (100 - x^2 - \dot{x}^2 + x - 3\dot{x} + 2x\dot{x})e^{2t}\, dt, \qquad x(0) = x_0, \quad x(1) = x_1.$$

(a) Find the Euler equation for this problem.

(b) Find the general solution of the equation you found in part (a).

Problem 4

Consider the following optimal control problem:

$$\max \int_0^T (2tx - 3u)\, dt, \quad \dot{x} = u, \quad x(0) = x_0, \quad x(T) = x_T, \quad u \in [0, 1],$$

where $x_0 < x_T < x_0 + T$.

(a) Write out the necessary conditions from the maximum principle for an admissible pair (x^*, u^*) to be optimal. (You can take for granted that $p_0 = 1$.)

(b) Solve the problem.

(c) Let $V(x_0, x_T, T) = \int_0^T (2tx^* - 3u^*)\, dt$ be the value function for the problem and let $H^*(T) = H(T, x^*(T), u^*(T), p(T))$ be the Hamiltonian evaluated at $(T, x^*(T), u^*(T), p(T))$. Verify that $\partial V / \partial T = H^*(T)$.
(x^* and u^* are the optimal functions that you found in part (b).)