Department of Economics
A. Strøm

Solutions of the examination problems in ECON4140/4145 Mathematics 3, 9 December 2009

Problem 1

(a) The symmetric matrix associated with Q is

$$\mathbf{A} = \begin{pmatrix} 1 & a & 1 \\ a & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

The leading principal minors of $\mathbf{A}$ are

$$D_1 = 1, \quad D_2 = \begin{vmatrix} 1 & a \\ a & 1 \end{vmatrix} = 1 - a^2, \quad D_3 = \begin{vmatrix} 1 & a & 1 \\ a & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = -(a-1)^2.$$

Since $D_3 \leq 0$ for all a, the quadratic form Q is not positive definite for any value of a.

(b) Q is positive semidefinite if and only if all principal minors of $\mathbf{A}$ are nonnegative. If $a \neq 1$, then $D_3 < 0$, so Q cannot be positive semidefinite. But if $a = 1$, then all minors of order 1 (not just the principal ones) are equal to 1 and all minors of order greater than 1 are 0 because all the rows of $\mathbf{A}$ are equal. So in this case Q is positive semidefinite.

(c) The characteristic polynomial of $\mathbf{A}$ is

$$\begin{aligned} p(\lambda) &= \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} \\ &= (1-\lambda)\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & 1-\lambda \end{vmatrix} + 1\begin{vmatrix} 1 & 1-\lambda \\ 1 & 1 \end{vmatrix} \\ &= -\lambda^3 + 3\lambda^2 = \lambda^2(3-\lambda) \end{aligned}$$

by cofactor expansion along the top row of the matrix. The eigenvalues of $\mathbf{A}$ are the roots of the equation $p(\lambda) = 0$, so we get

$$\lambda_1 = 3, \qquad \lambda_2 = \lambda_3 = 0.$$

Problem 2

(a) Equilibrium points are where $\dot{x} = \dot{y} = 0$, i.e. where

$$\text{(i)} \quad y = x, \qquad \text{(ii)} \quad y = -2x - \tfrac{1}{4}x^3.$$

m3xh09f 20.12.2009 1305

If we substitute x for y in (ii), we get

$$x = -2x - \tfrac{1}{4}x^3 \iff 3x + \tfrac{1}{4}x^3 = 0 \iff x(3 + \tfrac{1}{4}x^2) = 0\,.$$

The last equation obviously has only the solution $x = 0$, and therefore $(0,0)$ is the only equilibrium point for the given system.

The Jacobian matrix of the system at a point (x, y) is

$$\mathbf{J}(x,y) = \begin{pmatrix} -1 & 1 \\ 1 + \frac{3}{8}x^2 & \frac{1}{2} \end{pmatrix}.$$

In particular the Jacobian at $(0,0)$ is

$$\mathbf{A} = \mathbf{J}(0,0) = \begin{pmatrix} -1 & 1 \\ 1 & \frac{1}{2} \end{pmatrix}.$$

The determinant of $\mathbf{A}$ is $\begin{vmatrix} -1 & 1 \\ 1 & \frac{1}{2} \end{vmatrix} = -3/2 < 0$, so $(0,0)$ must be a saddle point.

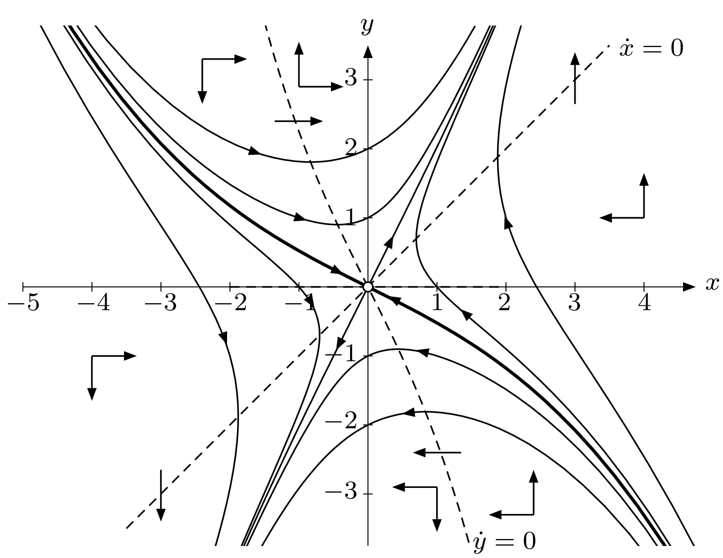
(b) The characteristic polynomial of $\mathbf{A}$ is

$$p(\lambda) = \begin{vmatrix} -1-\lambda & 1 \\ 1 & \frac{1}{2} - \lambda \end{vmatrix} = \lambda^2 + \tfrac{1}{2}\lambda - \tfrac{3}{2}\,.$$

The eigenvalues of $\mathbf{A}$ are the solutions of the equation $p(\lambda) = 0$, so we get $\lambda_1 = -3/2$ and $\lambda_2 = 1$. The eigenvectors $\mathbf{w} = (u, v)'$ corresponding to λ_1 are the nonzero solutions of $\mathbf{A}\mathbf{w} = -\frac{3}{2}\mathbf{w}$, that is,

$$\begin{aligned} -u + \phantom{\tfrac{1}{2}}v &= -\tfrac{3}{2}u \\ u + \tfrac{1}{2}v &= -\tfrac{3}{2}v \end{aligned} \iff \begin{aligned} \tfrac{1}{2}u + v &= 0 \\ u + 2v &= 0 \end{aligned} \iff u = -2v\,.$$

One such eigenvector is $(2, -1)'$.



m3xh09f 20.12.2009 1305

(c) The figure shows a phase diagram with some solution curves. The nullclines are shown as dashed curves, and the solutions converging to the equilibrium point are shown with heavier curves.

Problem 3

(a) With $F(t, x, \dot{x}) = (100 - x^2 - \dot{x}^2 + x - 3\dot{x} + 2x\dot{x})e^{2t}$ we get

$$F'_x(t, x, \dot{x}) = (-2x + 1 + 2\dot{x})e^{2t},$$
$$F'_{\dot{x}}(t, x, \dot{x}) = (-2\dot{x} - 3 + 2x)e^{2t},$$

and

$$\frac{d}{dt}\big(F'_{\dot{x}}(t, x, \dot{x})\big) = \cdots = (-2\ddot{x} - 2\dot{x} + 4x - 6)e^{2t}$$

The Euler equation is therefore

$$\begin{aligned} & F'_x(t, x, \dot{x}) - \frac{d}{dt}\big(F'_{\dot{x}}(t, x, \dot{x})\big) = 0 \\ \iff & -2x + 1 + 2\dot{x} - (-2\ddot{x} - 2\dot{x} + 4x - 6) = 0 \\ \iff & \ddot{x} + 2\dot{x} - 3x = -7/2 \qquad (*) \end{aligned}$$

(b) The characteristic equation is $r^2 + 2r - 3 = 0$, with the roots $r_1 = 1$, $r_2 = -3$. Thus the general solution of the homogeneous equation $\ddot{x} + 2\dot{x} - 3x = 0$ is $x = Ae^t + Be^{-3t}$, where A and B are arbitrary constants. To find the general solution of $(*)$ we need a particular solution u^*. Since the right-hand side is a constant we try with a constant function u^* and find $u^* = (-7/2)/(-3) = 7/6$. Hence, the general solution of $(*)$ is

$$x = Ae^t + Be^{-3t} + 7/6\,.$$

Problem 4

(a) The Hamiltonian is

$$H(t, x, u, p) = 2tx - 3u + pu = 2tx + (p - 3)u\,.$$

If (x^*, u^*) is an optimal pair there must exist a continuous and piecewise C^1 function p such that

(i) for each t in $[0, T]$, $u = u^*(t)$ maximizes

$$H(t, x^*(t), u, p(t)) = 2tx^*(t) + (p(t) - 3)u$$

for u in $[0, 1]$;

(ii) $\dot{p}(t) = -H'_x(t, x^*(t), u, p(t)) = -2t$ for each t in $[0, T]$ (except possibly where u^* is discontinuous).

There are no transversality conditions because the terminal value of x^* is given.

(b) Condition (ii) above implies that $\dot{p}(t) = -2t$ for all t in $[0, T]$ (even at points where u^* is discontinuous). Therefore

$$p(t) = A - t^2 \quad \text{for some constant } A.$$

Note that p is strictly decreasing.

From condition (i) we get

$$u^*(t) = \begin{cases} 1 & \text{if } p(t) > 3, \\ 0 & \text{if } p(t) < 3. \end{cases}$$

We cannot have $u^*(t) = 1$ for all t, because that would imply $x^*(t) \equiv x_0 + t$ and then $x^*(T) = x_0 + T > x_T$.

Similarly u^* cannot be identically 0, because then $x^*(t) \equiv x_0$ and we would have $x^*(T) = x_0 < x_T$.

It follows that there must exist a t^* in $(0, T)$ with $p(t^*) = 3$ and

$$u^*(t) = \begin{cases} 1 & \text{if } t \leq t^*, \\ 0 & \text{if } t > t^*. \end{cases}$$

Then

$$x^*(t) = \begin{cases} x_0 + t & \text{if } t \leq t^*, \\ x_0 + t^* & \text{if } t > t^*. \end{cases}$$

To determine t^* we use the condition $x^*(T) = x_T$, which gives $x_0 + t^* = x_T$, so

$$t^* = x_T - x_0\,.$$

Finally, the constant A in $p(t)$ is determined by $p(t^*) = A - (t^*)^2 = 3$, so $A = 3 + (t^*)^2$, and

$$p(t) = 3 + (t^*)^2 - t^2.$$

(c) The value function is

$$\begin{aligned} V(x_0, x_T, T) &= \int_0^T (2tx^*(t) - 3u^*(t))\,dt \\ &= \int_0^{t^*} (2t(x_0 + t) - 3)\,dt + \int_{t^*}^T 2tx_T\,dt \\ &= (\cdots) + \Big|_{t^*}^T x_T t^2 = (\cdots) + (T^2 - (t^*)^2)x_T\,. \end{aligned}$$

Note that t^* is independent of T. Therefore the integral over the interval $[0, t^*]$ is also independent of T and does not influence the partial derivative $\partial V/\partial T$. (A little calculation shows that the integral over the entire interval $[0, T]$ is equal to $-\frac{1}{3}(t^*)^3 - 3t^* + T^2 x_T$.) We get $\partial V/\partial T = 2Tx_T$.

Direct evaluation of H^* yields

$$\begin{aligned} H^*(T) = H(T, x^*(T), u^*(T), p(T)) &= 2Tx^*(T) - 3u^*(T) + p(T)u^*(T) \\ &= 2Tx_T - 0 + 0 = 2Tx_T. \end{aligned}$$

Thus, the equation $\partial V/\partial T = H^*(T)$ is verified.