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Solutions of the examination problems in ECON4140/4145 Mathematics 3, 15 December 2010

Problem 1

(a) Direct matrix multiplication shows that $\mathbf{A}\mathbf{v} = -\mathbf{v}$. Hence $\mathbf{v}$ is an eigenvector corresponding to the eigenvalue $\lambda_1 = -1$.

(b) The characteristic polynomial $p(\lambda) = |\mathbf{A} - \lambda\mathbf{I}|$ is a cubic polynomial with the eigenvalues of $\mathbf{A}$ as zeros. Since $p(-1) = 0$, the linear polynomial $\lambda - (-1) = \lambda + 1$ is a factor in $p(\lambda)$. The leading term in $p(\lambda)$ is $-\lambda^3$, so $p(\lambda) = -(\lambda+1)(\lambda^2 + a\lambda + b)$ for suitable a and b.

To find the values of a and b (which is not really necessary for this part of the problem) we calculate the characteristic polynomial

$$p(\lambda) = \begin{vmatrix} 1-\lambda & -2 & -2 \\ -1 & 1-\lambda & 0 \\ -1 & 0 & 1-\lambda \end{vmatrix} = -\lambda^3 + 3\lambda^2 + \lambda - 3\,,$$

for example by cofactor expansion along the bottom row or along the third column. Then

$$-\lambda^3 + 3\lambda^2 + \lambda - 3 = -(\lambda+1)(\lambda^2 + a\lambda + b) = -\lambda^3 - (a+1)\lambda^2 - (a+b)\lambda - b$$

which shows that $a = -4$ and $b = 3$.

(c) The eigenvalues are the roots of the equation $p(\lambda) = -(\lambda+1)(\lambda^2 - 4\lambda + 3) = 0$. We already know that $\lambda_1 = -1$ is an eigenvalue. The other eigenvalues are the roots of $\lambda^2 - 4\lambda + 3 = 0$, which are $\lambda_2 = 1$ and $\lambda_3 = 3$.

An eigenvector $\mathbf{w} = (x, y, z)'$ corresponding to an eigenvalue λ is a nontrivial solution of the equation system

$$\begin{aligned} (1-\lambda)x \; - \; 2y \; - \; 2z &= 0 \\ -x + (1-\lambda)y \qquad &= 0 \\ -x \qquad + (1-\lambda)z &= 0 \end{aligned}$$

For $\lambda = \lambda_2 = 1$ this system is equivalent to

$$\begin{aligned} -2y - 2z &= 0 \\ -x \qquad &= 0 \\ -x \qquad &= 0 \end{aligned} \quad \iff \quad x = 0,\ y = -z\,,$$

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and for $\lambda = \lambda_3 = 3$ we get

$$\begin{aligned} -2x - 2y - 2z &= 0 \\ -x - 2y \phantom{{}-2z} &= 0 \\ -x \phantom{{}-2y} - 2z &= 0 \end{aligned} \qquad \iff \qquad y = z,\ x = -2z\,.$$

Therefore eigenvectors corresponding to λ_2, λ_3 are $\mathbf{v}_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$ (or any nonzero multiples of these).

Problem 2

(a) The point $(2, 3)$ is an equilibrium point because $f(2, 3) = 0$ and $g(2, 3) = 0$. The Jacobian matrix of the system at a point (x, y) is $\mathbf{A}(x, y) = \begin{pmatrix} \frac{2}{3}x - \frac{4}{3} & 1 \\ -\frac{3}{2}x & -1 \end{pmatrix}$ and at $(2, 3)$ we get $\mathbf{A}(2, 3) = \begin{pmatrix} 0 & 1 \\ -3 & -1 \end{pmatrix}$. Since $|\mathbf{A}(2, 3)| = 3 > 0$ and $\mathrm{tr}(\mathbf{A}(2, 3)) = -1 < 0$, the equilibrium point $(2, 3)$ is locally asymptotically stable.

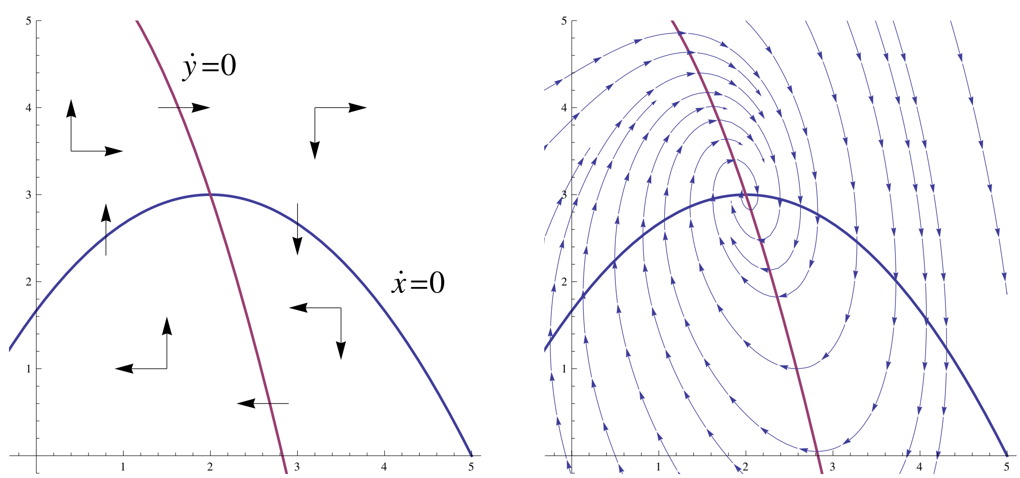
(b) A point (x, y) is an equilibrium point if and only if

$$\tfrac{1}{3}x^2 - \tfrac{4}{3}x - \tfrac{5}{3} + y = 0 \qquad (1)$$

$$6 - \tfrac{3}{4}x^2 - y = 0 \qquad (2)$$

If we add these equations we get $-\frac{5}{12}x^2 - \frac{4}{3}x + \frac{13}{3} = 0 \iff 5x^2 + 16x - 52 = 0$. This equation has the roots $x_1 = 2$ and $x_2 = -26/5$. By equation (2), the corresponding values of y are $y_1 = 3$ and $y_2 = -357/25$ $(= -14.28)$. (x_1, y_1) is the equilibrium that we know from before, and (x_2, y_2) is the only other equilibrium point.

(c) The two diagrams below show the nullclines and direction arrows (left) and the nullclines together with some solution curves (right).



Problem 3

(a) We write the constraint $x \geq 1$ in standard form as $-x \leq -1$. With the Lagrangian $\mathcal{L}(x, y) = ax + y - \lambda(x^2 + ay^2 - m) - \mu(-x + 1)$, the (Kuhn–Tucker) necessary conditions for (x^*, y^*) to be a solution of the problem are as follows: There must exist numbers λ and μ such that

$$\mathcal{L}'_1(x^*, y^*) = a - 2\lambda x^* + \mu = 0 \tag{1}$$
$$\mathcal{L}'_2(x^*, y^*) = 1 - 2a\lambda y^* = 0 \tag{2}$$
$$\lambda \geq 0 \quad (\lambda = 0 \text{ if } (x^*)^2 + a(y^*)^2 < m) \tag{3}$$
$$\mu \geq 0 \quad (\mu = 0 \text{ if } x^* > 1) \tag{4}$$
$$(x^*)^2 + a(y^*)^2 \leq m \tag{5}$$
$$x^* \geq 1 \tag{6}$$

The Lagrangian is obviously concave, so these conditions are also sufficient for optimality.

Condition (5) and (6) are the constraints in the problem. Since they are both given in terms of $\leq$-inequalities we know that the admissible set is closed. Condition (5) also shows that the admissible set is bounded, and it follows from the Extreme Value Theorem that the problem has a solution.

(b) Let $x^* = 1$, $y^* = \sqrt{(m-1)/a}$. Then (5) and (6) above are satisfied (with equality). Equations (1) and (2) give

$$a - 2\lambda + \mu = 0 \tag{1$'$}$$
$$1 - 2\lambda\sqrt{a(m-1)} = 0 \tag{2$'$}$$

From (2$'$) we get $2\lambda = 1/\sqrt{a(m-1)}$, and from (1$'$),

$$\mu = 2\lambda - a = \frac{1}{\sqrt{a(m-1)}} - a = \frac{1 - a\sqrt{a(m-1)}}{\sqrt{a(m-1)}}\,.$$

We see that

$$\mu \geq 0 \iff a\sqrt{a(m-1)} \leq 1 \iff a^3(m-1) \leq 1$$
$$\iff m - 1 \leq 1/a^3 \iff m \leq 1 + 1/a^3\,.$$

Thus, if $m \leq 1 + 1/a^3$, there exist nonnegative λ and μ that satisfy the Kuhn–Tucker conditions together with the given values of x^* and y^*.

If $m > 1 + 1/a^3$, then the given point (x^*, y^*) does not satisfy the Kuhn-Tucker conditions and is not optimal.

What other optimal points can there be? Condition (2) implies that λ cannot be 0. Therefore $\lambda > 0$ and y^* must also be positive. Since $\lambda > 0$, it follows from (3) that $(x^*)^2 + a(y^*)^2 = m$, and for a given x^*, y^* must be the nonnegative solution of this equation.

If $x^* = 1$ we get $y^* = \sqrt{(m-1)/a}$, as above.

But if $x^* > 1$, then $\mu = 0$ and equations (1) and (2) give $x^* = a/(2\lambda)$ and $y^* = 1/(2a\lambda)$. Then

$$m = (x^*)^2 + a(y^*)^2 = \frac{a^2}{4\lambda^2} + \frac{1}{4a\lambda^2} = \frac{a^3+1}{4a\lambda^2}\,.$$

It follows that

$$\lambda^2 = \frac{a^3+1}{4am}\,, \qquad \lambda = \frac{\sqrt{a^3+1}}{2\sqrt{am}}\,,$$

so

$$x^* = \frac{a}{2\lambda} = \frac{a\sqrt{am}}{\sqrt{a^3+1}} \qquad \text{and} \qquad y^* = \frac{1}{2a\lambda} = \frac{\sqrt{m}}{\sqrt{a(a^3+1)}}\,.$$

The constraint $x > 1$ now implies $a^3 m > a^3 + 1$, i.e. $m > 1 + 1/a^3$.

Conclusion: (I) If $m \le 1 + 1/a^3$, then

$$x^* = 1\,, \quad y^* = \sqrt{\frac{m-1}{a}}\,, \quad \lambda = \frac{1}{2\sqrt{a(m-1)}}\,, \quad \mu = \frac{1 - a\sqrt{a(m-1)}}{\sqrt{a(m-1)}}\,.$$

(II) If $m > 1 + 1/a^3$, then

$$x^* = \frac{a\sqrt{am}}{\sqrt{a^3+1}}\,, \quad y^* = \frac{\sqrt{m}}{\sqrt{a(a^3+1)}}\,, \quad \lambda = \frac{\sqrt{a^3+1}}{2\sqrt{am}}\,, \quad \mu = 0\,.$$

(If $m = 1 + 1/a^3$, the formulas in (II) give the same values for x^*, y^*, λ, and μ as the formulas in (I).)

(c) In case (I) above, $V(a,m) = ax^* + y^* = a + \sqrt{(m-1)/a}$, and we get

$$\frac{\partial V}{\partial m} = \frac{1}{\sqrt{a}}\,\frac{1}{2\sqrt{m-1}} = \frac{1}{2\sqrt{a(m-1)}} = \lambda\,.$$

In case (II),

$$\begin{aligned} V(a,m) = \frac{a^2\sqrt{am}}{\sqrt{a^3+1}} + \frac{\sqrt{m}}{\sqrt{a(a^3+1)}} &= \frac{a^2\sqrt{am}\sqrt{a}}{\sqrt{a(a^3+1)}} + \frac{\sqrt{m}}{\sqrt{a(a^3+1)}} \\ &= \frac{(a^3+1)\sqrt{m}}{\sqrt{a}\sqrt{a^3+1}} = \frac{\sqrt{a^3+1}\sqrt{m}}{\sqrt{a}} \end{aligned}$$

and

$$\frac{\partial V}{\partial m} = \frac{\sqrt{a^3+1}}{\sqrt{a}}\,\frac{1}{2\sqrt{m}} = \lambda\,.$$

(*Comment:* Strictly speaking, the calculation of $\partial V/\partial m$ above only holds when $m \ne m_0 = 1 + 1/a^3$. But since V is continuous with respect to m, and since the one-sided limits $\displaystyle\lim_{m\to(m_0)^-} \frac{\partial V}{\partial m}$ and $\displaystyle\lim_{m\to(m_0)^+} \frac{\partial V}{\partial m}$ exist and are equal, $\dfrac{\partial V}{\partial m}$ also exists when $m = m_0$.)

Problem 4

(a) The Hamiltonian is $H(t,x,u,p) = (x-u^2)e^{-t} + p(4ue^{-t} - x)$. If (x^*, u^*) is an optimal pair in this problem, there must exist a continuous and piecewise C^1 function p such that the following conditions are satisfied for each t in $[0,T]$:

(i) $u = u^*(t)$ must maximize $H(t, x^*(t), u, p(t))$ with respect to u, and therefore $H'(t, x^*(t), u, p(t)) = 0$, i.e. $-2u^*(t)e^{-t} + 4p(t)e^{-t} = 0$. Since H is concave with respect to u, this is also sufficient for maximization.

(ii) $\dot{p}(t) = -H'_x(t, x^*(t), u^*(t), p(t)) = -e^{-t} + p(t)$, with $p(T) = 0$ because $x(T)$ is free.

(iii) $\dot{x}^*(t) = 4u^*(t)e^{-t} - x^*(t)$, $x^*(0) = x_0$.

For each t in $[0,T]$ the Hamiltonian is concave in (x,u), so these conditions are sufficient for optimality.

(b) Condition (ii) implies that p must satisfy the differential equation $\dot{p} - p = -e^{-t}$. The general solution of this equation is $p = Ce^t + \frac{1}{2}e^{-t}$. (Use formula (5.4.4) in FMEA or formula (1.4.5) in MA2). Since we must have $p(T) = 0$, we get $Ce^T + \frac{1}{2}e^{-T} = 0$, so $C = -\frac{1}{2}e^{-2T}$ and

$$p(t) = \tfrac{1}{2}(e^{-t} - e^{t-2T}).$$

From (i) we get

$$u^*(t) = 2p(t) = e^{-t} - e^{t-2T},$$

and (iii) then gives the following differential equation for $x^*(t)$:

$$\dot{x}^*(t) + x^*(t) = 4(e^{-2t} - e^{-2T}).$$

The same formula from the textbook gives

$$x^*(t) = Ae^{-t} - 4e^{-2t} - 4e^{-2T}.$$

The initial condition $x^*(0) = x_0$ yields $x_0 = A - 4 - 4e^{-2T}$, so $A = x_0 + 4e^{-2T} + 4$ and

$$x^*(t) = (x_0 + 4 + 4e^{-2T})e^{-t} - 4e^{-2t} - 4e^{-2T}.$$

(c) The value function is

$$\begin{aligned}
V(x_0, T) &= \int_0^T \big(x^*(t) - (u^*(t))^2\big)e^{-t}\,dt \\
&= \int_0^T \big[(x_0 + 4 + 4e^{-2T})e^{-t} - 4e^{-2t} - 4e^{-2T} - (u^*(t))^2\big]e^{-t}\,dt \\
&= \int_0^T \big[x_0 e^{-2t} + \text{terms that do not depend on } x_0\big]\,dt.
\end{aligned}$$

By Leibniz's formula,

$$\frac{\partial V(x_0,T)}{\partial x_0} = \int_0^T \frac{\partial}{\partial x_0}(x_0 e^{-2t})\,dt = \int_0^T e^{-2t}\,dt$$
$$= \Big|_0^T -\tfrac{1}{2}e^{-2t} = \tfrac{1}{2}(1-e^{-2T}) = p(0)\,.$$

Alternatively, we could use

$$V(x_0,T) = \int_0^T x_0 e^{-2t} + \text{terms that do not depend on } x_0\big]\,dt$$
$$= x_0\int_0^T e^{-2t}\,dt + \text{something that does not depend on } x_0$$

and get

$$\frac{\partial V(x_0,T)}{\partial x_0} = \int_0^T e^{-2t}\,dt\,, \quad \text{etc. as above.}$$