University of Oslo / Department of Economics **English only**

ECON4140 Mathematics 3

Monday December 10 2012, 14:30–17:30

There are 2 pages of problems to be solved.

All printed and written material may be used, as well as pocket calculators.

Grades given run from A (best) to E for passes, and F for fail.

- You are required to state reasons for all your answers.
- You are permitted to use any information stated in an earlier letter-enumerated item (e.g. "(a)") to solve a later one (e.g. "(c)"), regardless of whether you managed to answer the former. A later item does not necessarily require answers from or information given in a previous one.

Problem 1 Define for each real number q the matrices $\mathbf{A}_q$ and $\mathbf{B}_q$ by

$$\mathbf{A}_q = \begin{pmatrix} q & 1 & 1 \\ 1 & q & q \\ 1 & q & q \end{pmatrix} \qquad \mathbf{B}_q = \begin{pmatrix} q & 1 & 1 & q \\ 1 & q & q & q \\ 1 & q & q & q \end{pmatrix} = (\mathbf{A}_q \vdots \mathbf{b}_q) \qquad \text{where } \mathbf{b}_q = q\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

(a) For each value of q, find the rank of $\mathbf{A}_q$ and the rank of $\mathbf{B}_q$ and decide whether the equation system $\mathbf{A}_q\mathbf{x} = \mathbf{b}_q$ has a solution. (Hint: can any row/column be deleted?)

(b) Explain why (or show that, if you prefer) $\mathbf{A}_q$ is neither positive definite nor negative definite, regardless of q.

(c) Let $\mathbf{u}_q = \begin{pmatrix} q \\ 0 \\ -1 \end{pmatrix}$ and $\mathbf{v}_q = \begin{pmatrix} q \\ -1 \\ 0 \end{pmatrix}$. Find those values of q for which $\mathbf{u}_q$ and/or $\mathbf{v}_q$ are eigenvectors of $\mathbf{A}_q$, and – when they are – the corresponding eigenvalue(s).

(d) Show that the following are eigenvalues for $\mathbf{A}_q$:

$$\frac{3q}{2} \pm \sqrt{(q/2)^2 + 2}$$

(e) Find an eigenvector $\mathbf{w}$ for $\mathbf{A}_q$ such that $\mathbf{w}$ does not depend on q.
(Hint: calculations from previous parts may be helpful; if you did not manage to solve completely parts (b) – (d), then you might alternatively look at part (a) for hints.)

Problem 2 Let $p \neq \pm\frac{1}{2}$ be a constant.

(a) Find the general solution of the difference equation

$$y_{t+2} - (2p+1)y_{t+1} + (p - \tfrac{1}{2})y_t = -t$$

(b) Find the general solution of the differential equation

$$\ddot{x}(t) - (2p+1)\dot{x}(t) + (p - \tfrac{1}{2})x(t) = -t \tag{D}$$

(c) Replace the right-hand side of (D) (the *differential* equation) by $-t - \sin t$.
Explain how to obtain a particular solution in this case.
(You are not required to carry out the calculations in full detail.)

Problem 3 Let $R > 0$ and $T > 0$, and consider the optimal control problem

$$\max \int_0^T \left(-\frac{(Rx)^2}{2} - \frac{u^R}{R} \right) dt, \qquad \text{where } \dot{x} = Rx - u, \quad u \in [0,1]$$
$$x(0) = \bar{x} \ (> 0), \qquad x(T) \text{ free}$$

Note: You are *not* asked to solve the optimal control problem (and you shouldn't try).

(a) (i) State the (necessary) conditions from the maximum principle.

(ii) For what values of $R > 0$ will a pair (x^*, u^*) that satisfies these conditions, solve the problem?

In the following, let $u^*(t)$ be an optimal control, so that u^* is a continuous function of t.

(b) Find $u^*(T)$.

(c) It turns out (and you are not supposed to prove) that the derivative of the value function V wrt. initial state $\bar{x}$, is negative, i.e.: $\partial V/\partial \bar{x} < 0$.
Use this to show that if $R > 1$, then $u^*(0)$ cannot be 0.
(Hint: With H being the Hamiltonian, what is $\partial H/\partial u$ when $u = 0$?)