ECON4140 Mathematics 3 – on the 2012–12–10 exam

This note is not suited as a complete solution or as a template for an exam paper. It was written as guidance for the grading committee. Errors (if any) are mine, not theirs.

Weighting of problems / letter items / questions to be decided at the grading committee's discretion. (Intention was that each letter item could be weighted approximately equal.)

Problem 1 Define for each real number q the matrices $\mathbf{A}_q$ and $\mathbf{B}_q$ by

$$\mathbf{A}_q = \begin{pmatrix} q & 1 & 1 \\ 1 & q & q \\ 1 & q & q \end{pmatrix} \qquad \mathbf{B}_q = \begin{pmatrix} q & 1 & 1 & q \\ 1 & q & q & q \\ 1 & q & q & q \end{pmatrix} = (\mathbf{A}_q \,\vdots\, \mathbf{b}_q) \qquad \text{where } \mathbf{b}_q = q\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

(a) For each value of q, find the rank of $\mathbf{A}_q$ and the rank of $\mathbf{B}_q$ and decide whether the equation system $\mathbf{A}_q\mathbf{x} = \mathbf{b}_q$ has a solution. (Hint: can any row/column be deleted?)

(b) Explain why (or show that, if you prefer) $\mathbf{A}_q$ is neither positive definite nor negative definite, regardless of q.

(c) Let $\mathbf{u}_q = \begin{pmatrix} q \\ 0 \\ -1 \end{pmatrix}$ and $\mathbf{v}_q = \begin{pmatrix} q \\ -1 \\ 0 \end{pmatrix}$. Find those values of q for which $\mathbf{u}_q$ and/or $\mathbf{v}_q$ are eigenvectors of $\mathbf{A}_q$, and – when they are – the corresponding eigenvalue(s).

(d) Show that the following are eigenvalues for $\mathbf{A}_q$:

$$\frac{3q}{2} \pm \sqrt{(q/2)^2 + 2}$$

(e) Find an eigenvector $\mathbf{w}$ for $\mathbf{A}_q$ such that $\mathbf{w}$ does not depend on q.
(Hint: calculations from previous parts may be helpful; if you did not manage to solve completely parts (b) – (d), then you might alternatively look at part (a) for hints.)

On the solution:

(a) For both matrices, the third row equals the second and can be deleted. (Also, the third column equals the second, and can be deleted as well.) The ranks are thus at most two. The leftmost 2×2 minor is $1 - q^2$, so:

if $q \neq \pm 1$, then rank($\mathbf{A}_q$) =rank($\mathbf{B}_q$) = 2 and the equation system has a solution.

If $q = 1$, then all elements are 1 and rank($\mathbf{A}_q$) =rank($\mathbf{B}_q$) = 1 and the equation system has a solution.

If $q = -1$, then the rows of $\mathbf{A}$ are proportional while the rightmost minor of $\mathbf{B}$ is -2; that is, $\underline{\underline{\text{rank}(\mathbf{A}_q) = 1, \text{rank}(\mathbf{B}_q) = 2 \text{ and the equation system has } \textit{no} \text{ solution}}}$.

(b) (The candidates are not required to point out that $\mathbf{A}_q$ is symmetric.)

From (a), $|\mathbf{A}_q| = 0$, violating necessary conditions for positive definiteness and for negative definiteness.

(c) $\mathbf{A}_q\mathbf{u}_q = \mathbf{A}_q\mathbf{v}_q = (q^2 - 1,\ 0,\ 0)^\top$, so in either case, the eigenvalue must be $\underline{\underline{= 0}}$, and we have the null vector if and only if $\underline{\underline{q = \pm 1}}$.

(d) By cofactor expansion, the characteristic polynomial is

$$(q-\lambda)[(q-\lambda)^2 - q^2] - 1 \cdot [q - \lambda - q] + 1 \cdot [q - q + \lambda] = \lambda\big[(q-\lambda)(\lambda - 2q) + 2\big)$$

so that there is one zero eigenvalue (known from part (a)!), and the other two are the zeroes of $-\lambda^2 + 3q\lambda + 2(1 - q^2)$, namely

$$-\frac{1}{2}\big[-3q \pm \sqrt{9q^2 + 4 \cdot 2 - 4q^2}\,\big] = \frac{3q}{2} \pm \sqrt{q^2/4 + 2}$$

which is what should be proven.

(e) From part (d) – or even from part (a) – we have a zero eigenvalue. Solving for an eigenvector, we can – again! – delete the third row of $\mathbf{A}_q$. It is clear that if $q = 0$, then $x_1 = 0$ and $x_2 = -x_3$, so this is necessary in order to hold for all q; by inspection, the vector $\underline{\underline{(0, -1, 1)^\top}}$ does the job.

Problem 2 Let $p \neq \pm\frac{1}{2}$ be a constant.

(a) Find the general solution of the difference equation

$$y_{t+2} - (2p+1)y_{t+1} + (p - \tfrac{1}{2})y_t = -t$$

(b) Find the general solution of the differential equation

$$\ddot{x}(t) - (2p+1)\dot{x}(t) + (p - \tfrac{1}{2})x(t) = -t \qquad \text{(D)}$$

(c) Replace the right-hand side of (D) (the *differential* equation) by $-t - \sin t$.
Explain how to obtain a particular solution in this case.
(You are not required to carry out the calculations in full detail.)

On the solution: For both part (a) and (b), the characteristic equation is

$$r^2 - (2p+1)r + (p - 1/2) = 0$$

with two real roots

$$r_\pm = \frac{1}{2}\left[2p + 1 \pm \sqrt{4p^2 + 4p + 1 - 4p + 2}\,\right] = p + \frac{1}{2} \pm \sqrt{p^2 + 3/4}$$

(a) We look for a linear particular solution of the form $at + b$:

$$a(t+2) + b - (2p+1)[a(t+1) + b] + (p - 1/2)[at + b] = -t$$

Equating first-order coefficients, we get $-t = at(1-2p-1+p-1/2) = -at(p+1/2)$, so that $a = (p+1/2)^{-1}$. The constant terms must vanish: $0 = 2a + b - (2p+1)(a + b) + (p - 1/2)b$ so that $b = (1-2p)a/(p+1/2)$. The general solution is therefore

$$\underline{\underline{A\big(p + \tfrac{1}{2} + \sqrt{p^2 + \tfrac{3}{4}}\big)^t + B\big(p + \tfrac{1}{2} - \sqrt{p^2 + \tfrac{3}{4}}\big)^t + \frac{t}{p + 1/2} + \frac{1 - 2p}{(p + \frac{1}{2})^2}}}$$

(b) For a particular solution of the differential equation, we again look for a linear form $at + b$:

$$-(2p+1)a + (p - 1/2)(at + b) = -t$$

so that $a = -1/(p-1/2)$ and $b = (2p+1)a/(p-1/2) = -(2p+1)/(p-1/2)^2$. The general solution is therefore

$$\underline{\underline{A \exp\{\big(p + \tfrac{1}{2} + \sqrt{p^2 + \tfrac{3}{4}}\big)t\} + B \exp\{\big(p + \tfrac{1}{2} - \sqrt{p^2 + \tfrac{3}{4}}\big)t\} - \frac{t}{p - \frac{1}{2}} - \frac{2p + 1}{(p - \frac{1}{2})^2}}}$$

(c) Suffices to point out: there will be an additional term of the form $K \sin t + L \cos t$.

(Note: some candidates will likely write e.g. «put $u^* = K \sin t + L \cos t$» etc. The intention of the problem is to test whether they get both the sin and the cos term, and sloppiness on whether to keep or replace the linear contribution, should not be penalized harshly.)

Problem 3 Let $R > 0$ and $T > 0$, and consider the optimal control problem

$$\max \int_0^T \left(-\frac{(Rx)^2}{2} - \frac{u^R}{R} \right) dt, \qquad \text{where } \dot{x} = Rx - u, \quad u \in [0,1]$$
$$x(0) = \bar{x} \ (> 0), \qquad x(T) \text{ free}$$

Note: You are *not* asked to solve the optimal control problem (and you shouldn't try).

(a) (i) State the (necessary) conditions from the maximum principle.

(ii) For what values of $R > 0$ will a pair (x^*, u^*) that satisfies these conditions, solve the problem?

In the following, let $u^*(t)$ be an optimal control, so that u^* is a continuous function of t.

(b) Find $u^*(T)$.

(c) It turns out (and you are not supposed to prove) that the derivative of the value function V wrt. initial state $\bar{x}$, is negative, i.e.: $\partial V/\partial \bar{x} < 0$.
Use this to show that if $R > 1$, then $u^*(0)$ cannot be 0.
(Hint: With H being the Hamiltonian, what is $\partial H/\partial u$ when $u = 0$?)

On the solution

(a) Define the Hamiltonian as $H(t, x, u, p) = -\frac{1}{2}(Rx)^2 - \frac{1}{R}u^R + p(Rx - u)$.

(i) For optimum, we must necessarily have the following:

- $u^*(t)$ maximizing H for $u \in [0,1]$.

  Note: The candidates' answers may vary in level of detail here, but this should do; the applications of the maximization are tested in parts (b) and to some degree part (c).

  We do have, though, that if $p \geq 0$ then $u^* = 0$; if $R \leq 1$ then by convexity $u^* = 1$ if $(-p) > 1/R$ and $= 0$ if $< 1/R$; if $R > 1$ then $u^* = (-p)^{1/(R-1)}$ for $p \in [-1, 0)$ and 1 for $p < -1$.
- $\dot{p} = -H'_x = R(Rx - p)$ with $p(T) = 0$;
- $\dot{x} = Rx - u$ (with $x(0) = \bar{x}$, though that need likely not be mentioned).

(ii) For $R \geq 1$, we have u^R convex, and then $(x, u) \mapsto H$ is concave, being a sum of concaves. The Magnasarian sufficient conditions then apply. (For $R < 1$, this fails.)

Note: The Arrow condition has not been so much stressed in seminar problems that it can be expected covered, and it will be sufficient for the score of an

«A» on this part (a) to cover (flawlessly) the necessary conditions in (i) and Mangasarian in (ii). However, it will certainly not be held against a candidate if (s)he points out that $H = -\frac{1}{R}u^R - pu$ plus something which does not depend on u but is concave in x, so that the maximized Hamiltonian is concave in x, and the Arrow condition applies – for *any* $R > 0$.

(b) $p(T) = 0$, so $\underline{\underline{u^*(T) = 0}}$ maximizes $H(T, x, u, p) = -\frac{1}{2}(Rx)^2 - \frac{1}{R}u^R$.

(c) We have $p(0) = \partial V/\partial \bar{x}$, which is < 0. Since $\partial H/\partial u = -u^{R-1} - p$, then for $R > 1$ we have $\partial H/\partial u|_{u=0} = -u^{R-1}|_{u=0} - p = -p > 0$, so there cannot be maximum for $u = 0$.