University of Oslo / Department of Economics **English only**

ECON4140 Mathematics 3

December 16th 2013, 0900–1200.

There are 2 pages of problems to be solved.

All printed and written material may be used, as well as pocket calculators.

Grades given run from A (best) to E for passes, and F for fail.

- You are required to state reasons for all your answers.
- You are permitted to use any information stated in an earlier item (e.g. "(a)") to solve a later one (e.g. "(c)"), regardless of whether you managed to answer the former. A later item does not necessarily require answers from or information given in a previous one.
- You can use English, Norwegian, Swedish or Danish language.

Problem 1 For each 2×2 matrix $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, let $\mathbf{F}$ be the 4×4 matrix $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & c & d & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

(a). Consider the quadratic form $Q(w,x,y,z) = (w,\ x,\ y,\ z)\ \mathbf{F}\ (w,\ x,\ y,\ z)'$.
Find $\mathbf{A}$ such that Q is positive definite, or show that no such $\mathbf{A}$ exists.

(b). (i) Will $\mathbf{A}$ always have a (real) eigenvalue?

(ii) Will $\mathbf{F}$ always have a (real) eigenvalue?

In the following, let $\mathbf{A} = \begin{pmatrix} 17 & 12 \\ 12 & 7 \end{pmatrix}$.

(c). (i) Show that $\mathbf{v} = (2,\ -3)'$ is an eigenvector for $\mathbf{A}$ with negative eigenvalue λ, and

(ii) find p and q such that $(p,\ 2,\ -3,\ q)'$ is an eigenvector for $\mathbf{F}$.

(d). Find an eigenvalue κ for $\mathbf{A}$, with $\kappa \neq \lambda$, and an associated eigenvector $\mathbf{u}$.

(e). The origin is an equilibrium point for the differential equation system

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \mathbf{A} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + (x(t) + y(t))^{2013} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Decide whether the origin is stable, or a saddle point, or neither; or show that the tools of this course do not yield conclusion.

Problem 2 Consider the differential equation

$$\ddot{x}(t) + \dot{x}(t) + x(t) = f(t) \tag{E}$$

(a). Find the general solution of (E) for the case $f = 37$.

(b). Explain how you would go forth to find the general solution of (E) for the case where $f(t) = t + 4e^t + 5\sin(9t)$. (You are not asked to do the full detail.)

Problem 3 Consider the dynamic programming problem

$$J_{t_0}(x) = \max_{u_t \geq 0} \left\{ \sum_{t=t_0}^{T-1} (-x_t u_t) + x_T \right\} \quad \text{subject to} \quad x_{t+1} = x_t - e^{-u_t}, \quad x_{t_0} = x \geq T - t_0$$

Observe that since $x_{t_0} \geq T - t_0$ and $e^{-u_t} \leq 1$, then $x_t \geq 1$ for all $t \leq T - 1$.

Use this to show by induction that

$$J_t(x) = x + t - T \quad \text{with} \quad u_t = 0 \text{ being optimal}$$

is true for all $x_0 \geq T - t_0$ and all $t = t_0, \ldots, T$.

(If you are unable to do this, calculate instead J_{T-1} and J_{T-2} for up to 50 percent score.)

Problem 4 Let $U = [0, 1]$ and consider the optimal control problem

$$\max_{u(t) \in U} \int_0^5 \left[e^{u(t)} + x(t) \right] dt \quad \text{subject to} \quad \dot{x}(t) = \ln\big(2 - u(t)\big), \quad x(0) = x_0, \quad x(5) \text{ free.}$$

(You are *not* asked to solve this optimal control problem!)

(a). State the necessary conditions from the maximum principle.

(b). Show that for some t the optimal control must be the maximum allowed value $u = 1$.

(c). Assume there is a pair (x^*, u^*) that satisfies the necessary conditions from the maximum principle. Show that this pair will solve the problem.