University of Oslo / Department of Economics / NCF

ECON4140 Mathematics 3 – on the 2013 exam

About this document. Notes like these are usually guidance for the grading committee, and have taken various formats over the years – sometimes a suggested exam paper (with or without additional comments and annotations), sometimes a note more useful to the committee and not at all suited for an exam paper template. This year I have taken a different approach, due to questions from students on how much they «have to» write; I have (hand-) written a (nearly) «minimal» solution. This (typeset) part of the document will supplement the solution note with a few comments on what is to be expected.

About weighting and grading. The exam was written with the *intention* to facilitate uniform weighting of the letter-enumerated items if the committee finds that appropriate; no weighting was specified and thus none committed to, in case the committee finds a different weighting appropriate. In the event of appeals, the appeals committe will choose weighting at their discretion.

About the attached solution note. This is (with a couple of reservations to be remarked upon below) nearly minimal. It is not intended as a recommended format or template for an exam paper – indeed, it likely is not to be recommended, as the deductions are minimally documented, leaving the paper vulnerable to errors; in case of errors, including more elaborate calculations and reasoning would be likely to allow for partial score, as it documents how far the candidate did correct, and whether the erroneous part is due to trivial and less interesting errors or to wrong theory.
And I certainly do not recomment trying to fit an exam paper on least possible sheets of paper.

Remarks to problem 1:

(a). The note just gives a counterexample; it is of course also OK to point out that a(ny other) necessary condition for positive definiteness fails.

(c). One may argue that from $\mathbf{Av} = -\mathbf{v}$ it is selfevident that the eigenvalue is negative. Maybe the note isn't «minimal» on this point.

(d). The solution note could arguably be shorter, due to the fact that eigenvalues (counted with algebraic multiplicity) sum up to trace, $\Rightarrow \kappa = 17 + 7 - \lambda = 25$ – that suffices as answer, and also will $\kappa = |\mathbf{A}|/\lambda = -(109 - 144) = 25$. For the note I chose the method that has been stressed in class.

Remarks to problem 2: The note uses the $C\cos(\omega+\beta t)$ form, and the candidates may use the $A\sin + B\cos$ form at their discretion. The intention for part (b) was to check if they know they need both the constant and – most important – both a sin and a cos term, or both C and ω. (It is so evident that the first-order coefficient will be 1, that it likely goes without saying, i.e. no argument is needed for that.)

Remarks to problem 3: This problem tests whether they know an induction proof, and whether they know the dynamic programming equation; the «up to 50 percent score» for merely the latter, reflects that. It is not intended to nitpick on the final u_T^* which does not enter the problem.

Remarks to problem 4:

(a).
- One may argue that the «(i.e. [. . .])» part on the note is redundant for part (a). Indeed the wording of part (a) does not require finding p from the differential equations; therefore, the candidates cannot be expected to know whether u^* is stationary or endpoint, and thus there is no straightforward such condition to state at the stage of part (a) (although it is not hard to see, and will not be held against a candidate who makes merely an incomplete argument in the direction of the following: $p \geq 0$ thus by convexity u^* is endpoint thus u^* is zero when $p\ln 2 > e-1$ and one when $p\ln 2 < e-1$).
- One may argue that the $\dot{x}$ differential equation goes without saying (just like the literature usually omits admissibility from the «Kuhn–Tucker» conditions).
- The transversality condition must be included though.

(b). The candidates are allowed to disregard the fact that single points do not contribute to the integral, and they are thus not expected to make any argument of the type «therefore, for all t sufficently close to 5». (The note does not make such an argument either.)

(c). Here, Arrow's condition is *the* piece of theory to apply (though it doesn't have to be mentioned by name). Comparing with the 2012 solution note, where Arrow was said to be less crucial, the following were made explicit to the students upon reviewing the 2012 exam in lecture:

- Arrow being less essential in 2012 was justified by that condition not having been so much focused in 2012 – as pointed out in that note. This year was different.
- It was in the 2012 exam also due to Mangasarian applying to «most» R values – which makes it easier to forget to check Arrow when Mangasarian fails. For the 2013 exam Mangasarian is not useful, and the wording is «show that» – then they *are* expected to find the condition which establishes what they are asked to show.

ECON 4140 Exam 2013 - 12 - 16 : a "minimal" solution

1

(a) Impossible, as $(1\ 0\ 0\ 0)\,\mathbb{F}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = 0$

(b) i) **No**. Eigenvalues where $\lambda^2 - (\operatorname{tr} A)\lambda + \det A = 0$,

not real if $-a_{12}a_{21}$ is large.

ii) **Yes**. Zero, by (a)

(c) i) $A\mathbf{v} = \begin{pmatrix}34-36\\24-21\end{pmatrix} = -\mathbf{v}$, eigenvalue $= -1$. (<0)

ii) $\mathbb{F}\begin{pmatrix}p\\ \mathbf{v}\\ q\end{pmatrix} = \begin{pmatrix}0\\ -\mathbf{v}\\ 0\end{pmatrix}$, Ok with $\underline{p=q=0}$ (eigenv. $=-1$)

(d) $\frac{\operatorname{tr} A}{2} \pm \sqrt{\left(\frac{\operatorname{tr} A}{2}\right)^2 - \det A} = 12 \pm \sqrt{144 - 119 + 144} = 12 \pm 13.$

$\lambda = -1$, so $\kappa = \underline{25}$

$\left(\begin{array}{cc|c}-13 & 12 & 0\\ 12 & -18 & 0\end{array}\right)$ yields $\underline{\mathbf{u} = \begin{pmatrix}3\\2\end{pmatrix}}$

(e) Jacobian $= A$ (as $(x+y)^{2013}$ has Jacobian $= 0$)

Eigenvalues $\kappa > 0 > \lambda \Rightarrow$ **saddle point**

2

$r^2 + r + 1 = 0 \iff r = -\frac{1}{2} \pm \frac{1}{2}\sqrt{1-4}$, not real.

(a) $\underline{x(t) = Ce^{-t/2}\cos\left(\omega + \frac{t}{2}\sqrt{3}\right) + u^*}$, where

$u^* = $ constant: $\cancel{\ddot{u}^*} + \cancel{\dot{u}^*} + 1u^* = 37$, $\underline{u^* = 37}$

(b) Particular solution $v^* = at + b + De^{t} + K\sin 9t + L\cos 9t$

(then fit constants)

3

$J_T = x + T - T = x$, $\underline{OK}$ (u_T^* arbitrary)

For induction, suppose Ok at t. Then at $t-1$:

$$J_{t-1} = \max_{u\ge 0}\left\{-xu + x - e^{-u} + t - T\right\}$$

u-derivative: $e^{-u} - x \le 0$ (as $e^{-u} \le 1 \le x_t$)

So $\underline{u^*_{t-1} = 0}$ and $J_{t-1} = 0 + x - 1 + t - T$, $\underline{OK}$.

4 $H(t, x, u, p) = e^{u} + x + p\cdot\ln(2-u)$

(a) Conditions:
* u^* maximizes H over $u \in [0,1]$ (i.e. u^* maximizes $e^{u} + p\ln(2-u)$)
* $\dot{p} = -\frac{\partial H}{\partial x} = -1$ with $\underline{p(5) = 0}$
* $\dot{x}^* = \ln(2 - u^*)$

(c) As u^* maximizes $\underline{e^{u} + p\ln(2-u)}$ (no x), then

$u^* = \hat{u}(p)$ and

$H(t, x, \hat{u}(p), p) = x + \text{some } w(p)$

is $\underline{\text{concave w.r.t. } x}$ (Arrow)

(b) At $t = 5$, $p(5) = 0$ and $u^*(5)$ maximizes e^{u}. Choose $\underline{\text{maximal } u^* = 1}$