University of Oslo / Department of Economics ***(corrected version, see grading guideline)***

ECON4140 Mathematics 3

December 17th 2014, 0900–1200.

There are 2 pages of problems to be solved.

All printed and written material may be used, as well as pocket calculators.

Grades given run from A (best) to E for passes, and F for fail.

- You are required to state reasons for all your answers.
- You are permitted to use any information stated in an earlier item (e.g. "(a)") to solve a later one (e.g. "(c)"), regardless of whether you managed to answer the former. A later item does not necessarily require answers from or information given in a previous one.

Problem 1

(a) Evaluate $\int_{-\pi}^{\pi} \Big(\int_{\pi}^{2\pi} \frac{\sin(xy)}{x}\, dx \Big)\, dy.$ (*Hint:* You will need a symmetry property.)

(b) Let $f(x)$ be a given C^2 strictly increasing strictly convex function defined for all real x, and define f_1, f_2, ... inductively by

$$f_1(x) = f(x), \qquad \text{and} \qquad f_{n+1}(x) = f'(f_n(x)), \quad \text{each } n = 1, 2, \dots$$

Use induction to show that all the f_n are quasiconvex.

Note: Information from problem 2 will be used in problem 3, and information from problem 3 in problem 4. The «bonus» status of problem 4 part (c) is due to this dependence.

Problem 2

Let $m > 0$ be a constant. Consider for each m the matrices

$$\mathbf{A} = \mathbf{A}_m = \begin{pmatrix} m^3 & \frac{3}{2}m^{-7} \\ \frac{1}{2}m^{13} & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \mathbf{B}_m = \begin{pmatrix} m^3 & \frac{3}{2}m^{-7} & 2 \\ \frac{1}{2}m^{13} & 0 & 0 \\ m^5 & m^{-5} & 4 \end{pmatrix}$$

(a) Show that $(-m^{-3},\, m^7\,)'$ is an eigenvector for $\mathbf{A}_m$, and that its associated eigenvalue $\lambda = \lambda(m)$ is negative.

(b) Find the other eigenvalue $\mu = \mu(m)$ of $\mathbf{A}_m$, and an associated eigenvector.

(c) Find the only $m > 0$ such that $\mathbf{B}_m$ and $\mathbf{A}_m$ have same rank.

Problem 3 Let G and H be C^2 functions defined on $(0, \infty)$, let $m > 0$ be a constant and S be the open first quadrant $S = \{(x, y);\ x > 0,\ y > 0\}$. For $x = x(t)$, $y = y(t)$, consider the differential equation system – valid from time $t = 0$ until the first time $T \geq 0$ for which $(x(T), y(T)) \notin S$:

$$\begin{aligned} \dot{x} &= G(x) + H(y) \\ \dot{y} &= \big[m^3 - G'(x)\big] \cdot y \end{aligned} \tag{D}$$

(Observe that there is a derivative sign «G'» in the second equation.)

(a) Show that if $H' > 0 > G''$ (so that in particular, $m^3 - G'$ is strictly increasing), then
  (i) the system has at most one equilibrium point in S (note $xy > 0$ in S!), and
  (ii) if such one exists, it is a saddle point. (*Hint:* a term will vanish and simplify.)

Let from now on $G(x) = 2x^{1/2}$ and $H(y) = -2y^{-3/4}$ so that the saddle point has coordinates $(\bar{x}, \bar{y}) = (m^{-6}, m^4)$. (You need not show this.)

(b) Put $m = 1$. For those two integral curves (i.e. particular solution trajectories) $(x(t), y(t))$ which converge to $(\bar{x}, \bar{y})$ as $t \to +\infty$, show that the *slope* $\frac{y(t)-\bar{y}}{x(t)-\bar{x}}$ converges to -1. (*Hint:* Problem 2 gives information which likely saves time.)

(c) Put $m = 1$. Sketch a phase diagram and indicate a few representative integral curves.

Problem 4 Let $x_0 > 0$ and consider – but do not solve! – the optimal control problem

$$V(x_0) = \max_{u(t) \geq 0} \int_0^{2014} \frac{-6e^{-t}}{u(t)}\, dt, \quad \text{where} \quad x(0) = x_0, \quad x(2014) \geq 0, \qquad \dot{x} = 2x^{1/2} - 2u^3.$$

(a) State the conditions from the maximum principle.
(You can safely disregard the «p_0» constant and put it $= 1$).

(b) Let $x(t)$ satisfy the conditions from the maximum principle with adjoint variable $p(t)$. Let $y(t) = e^t p(t)$, so that $\dot{y} = y + e^t \dot{p}$ (then y is the current-value adjoint).
Show that (x, y) satisfies the differential equation system (D) of Problem 3, with $G(x) = 2x^{1/2}$, $H(y) = -2y^{-3/4}$ and $m = 1$ (as in Problem 3 part (c)).
(*Hint:* you shall obtain the condition $u(t) = \big(y(t)\big)^{-1/4}$.)

(c) *«Bonus» question: this part will be deleted (zero-weighted) if that benefits your grade.*
Consider your phase diagram for Problem 3 part (c), and assume $x(0) = x_0 = 1 = \bar{x}$ (the x-coordinate of the saddle point). Take for granted that the optimal path x^* ends at $x^*(2014) = 0$. Use this to argue for an upper *or* a lower bound for $V'(1)$; i.e.,

Find an appropriate $a > 0$ and
- either argue that $V'(1) \leq a$
- or argue that $V'(1) \geq a$.

(Recall that $V = V(x_0)$ is the optimal value as function of initial state x_0.)