University of Oslo / Department of Economics / NCF

ECON4140 Mathematics 3 – on the 2014 exam

Note on the exam set, problem 3 (b). During the exam it was announced a correction to problem 3 part (b). See the specific remarks below.

About this document. Notes like these are usually guidance for the grading committee, and have taken various formats over the years – sometimes a suggested exam paper (with or without additional comments and annotations), sometimes a note more useful to the committee and not at all suited for an exam paper template. Last year I responded to questions from students on how much they «have to» write by producing a (nearly) «minimal» solution. This year I have produced a handwritten solution which should be *closer to* what an exam paper could look like[1].

About weighting and grading. There was a special «bonus» provision for problem 4 (c), to be interpreted as follows: each paper should be assessed (I) with all problems and answers, and (II) as if problem 4 (c) did not exist (and neither the answer), so that e.g. a perfect score on all other problems is 100 percent score on the problem set. Among scores/grades (I) and (II), the best shall be chosen. The committee has discretion to consider whether this rule had unreasonable effects, but the way the problem set is worded, adjustments should not reduce scores from the stipulated $\max\{\text{score}_I, \text{score}_{II}\}$.

Apart from that, the exam was written with the *intention* to facilitate uniform weighting of the letter-enumerated items if the committee finds that appropriate; no weighting was specified and thus none committed to, in case the committee finds a different weighting appropriate. In the event of appeals, the appeals committe will choose weighting at their discretion.

Remarks to problem 1:

(a). The problem has three elements; 1, changing the order of integration; 2, being able to carry out an integration where a variable is treated as a constant; 3, the symmetry of the cosine function.

(b). The problem involves both induction and criteria for quasiconvexity. *Intention* – not binding the exam commitee – was that these elements be at last approximately equally important.

The note writes it more explicite than necessary that each f_n is a composition as stated. A wording addressing quasiconvexity more «directly» is, I assume, likely.

[1] which is the reason why each problem starts a new page – that is often to be recommended

Remarks to problem 2: Part (a) is solved by multiply-and-identify. It is of course OK to solve by applying determinant to find both eigenvalues etc., and boil out (a) and (b) simultaneously. The note uses the shortcut that some equation is superfluous – it is easy here to see that *any* is. The rank question is straightforward.

Remarks to problem 3 (a) and (c):

(a). Item (i) uses merely one-to-one functions. Item (ii) requires the Jacobian – it suffices to point out that the determinant is negative.

(c). The wording is deliberately «sketch», not «draw». I assume a sketch should include nullclines of the right shape, the correct vertical/horizontal crossings, NW/NE/SE/SW arrows and a few trajectories. It is not asked explicitly to include the convergent ones, but it is certainly no disadvantage.

Remarks to problem 3 (b): This had an unfortunate error. The intention was that the Jacobian should be identical to $\mathbf{A}_m$ from problem 2 so that one could use the slope of the eigenvector *stated in the problem* – i.e. it was not the intention that one should need to have *done* problem 2, only to look up what was written.

Unfortunately the matrices are not equal and the result to show wrong[2]. During the exam, it was announced that as a fixup they should assume $m = 1$ for part 3 (b) as well. The committee should apply their best judgement to take into account that

- there was an error, and the change was announced only at 1105 with less than an hour left, when the error could have cost time already;
- even with the correction, when the statement as well as the connection to problem 2 is valid, the hint is not equally helpful as intended; cf. the footnote, it was intentional that the Jacobian should not only be equal to $\mathbf{A}_1$ (for which one would have to insert $m = 1$) but even match the precise formula on page 1 – this to make the problem easier.

According to the invilgators, none of the ECON4140 candidate were absent during the announcement, and in particular, none had yet submitted their paper.

[2]It is of less importance why and how the error occured; at one stage in the drafts, $m = 1$ was assumed already in 3 (b), but I reverted to general m in order to have the formula looking precisely like $\mathbf{A}_m$, making it easier to recognize than $\begin{pmatrix} 1 & 1.5 \\ 0.5 & 0 \end{pmatrix}$ being equal to $\mathbf{A}_1$. In the meantime, a power had however been tweaked to make the calculations more tidy.

Remarks to problem 4 (a) and (b): For parts (a) and (b) there are arguably elements which «go without saying». In particular, the intention of part (b) was to test ability to extract information from the maximum principle and apply it: the differential equation for p and the maximality of u^* – not to elaborate on why $u \mapsto H$ has internal max, so simple application of the first-order condition is sufficient.

Remarks to problem 4 (c): The intention was to identify $V'(x_0) = p(0)$. Noticing that $y(0) = e^0 p(0)$ cannot start above the saddle path is of course key to the argument why, but a paper that manages to point out that the question must be related to the fact that $V'(x_0) = p(0)$ and hopefully say something sensible that reveals understanding that (x, p) or (x, y) satisfy a differential equation (system) in $\mathbf{R}^2$, should be generously awarded for picking up the main point of the problem.

For how to apply the «bonus» status, see the introductory remarks; the reason for this status was – in addition to making it slightly easier to get a high score by choosing max of two calculations – that it does to a great extent depend on information from the *solution of* a previous problem. (Not like the intention of the misgiven hint to problem 3 (b), where only the information from the problem *statement* would – if not for the error – suffice.)

Exam ECON 4140, 2014-12-17

Possible solution.

1 (a)

$$\int_{-\pi}^{\pi}\int_{\pi}^{2\pi}\frac{\sin(xy)}{x}\,dx\,dy = \int_{\pi}^{2\pi}\int_{-\pi}^{\pi}\frac{\sin(xy)}{x}\,dy\,dx$$

$$= \int_{\pi}^{2\pi}\underbrace{\left[-\frac{\cos(xy)}{x^2}\right]_{y=-\pi}^{y=\pi}}_{=0 \text{ since } \cos\theta = \cos(-\theta)}dx \quad = \underline{\underline{0}}$$

1 (b)

f strictly convex so f' strictly increasing

Observe: $f_2 = f'(f(x))$

strictly increasing — str. convex ergo quasiconvex.

Suppose for induction that

$$f_n(x) = g_n(f(x))$$

where g_n nondecreasing.

Then $f_{n+1}(x) = f'(g_n(f(x)))$

$$= g_{n+1}(f(x))$$

where $g_{n+1}(x) = f'(g_n(x))$

nondecreasing

is nondecreasing, <u>OK</u>.

→ OK for $n=1$ with $g_1(x) = x$.

So f_n is a nondecreasing function of a convex function of x, ergo quasiconvex.

2 (a) $$\bar{A}_m \begin{pmatrix} -m^{-3} \\ m^{7} \end{pmatrix} = \begin{pmatrix} -1 + \frac{3}{2} \\ -\frac{1}{2} m^{10} \end{pmatrix} = -\frac{m^3}{2} \begin{pmatrix} -m^{-3} \\ m^{7} \end{pmatrix}$$

so $\lambda(m) = -\frac{1}{2} m^3 < 0$

(as $m > 0$)

2018: Fixed negative signs and the "2" denominator.

2 (b) We have $|\bar{A}_m - \mu \bar{I}|$

$$= (m^3 - \mu)(-\mu) - \frac{3}{4} m^6$$

$$= \mu^2 - m^3 \mu - \frac{3}{4} m^6 .$$

$$\mu = \frac{1}{2}\left[m^3 + \sqrt{m^6 + 3m^6} \right]$$

(as "$-$" yields λ)

$$= \frac{m^3}{2}\left[1 + \sqrt{4} \right] = \underline{\underline{\frac{3}{2} m^3}} .$$

One eq superfluous. solve

$$\begin{pmatrix} \frac{1}{2} m^{13} & -\frac{3}{2} m^3 \end{pmatrix} \bar{v} = 0 :$$

$$\underline{\underline{\bar{v} = \begin{pmatrix} 3 \\ m^{10} \end{pmatrix}}}$$

is an eigenvector associated to μ.

[And so is every nonzero scaling]

2 (c)

$$\det(\bar{A}_m) = \frac{3}{4} m^6 \neq 0 \qquad (\text{as } m > 0)$$

so $\operatorname{rank}(\bar{A}_m) = 2$.

$$\bar{B}_m = \begin{pmatrix} \bar{A}_m & \begin{matrix} b_1 \\ b_2 \end{matrix} \\ b_5 \ b_4 & b_3 \end{pmatrix} \quad \text{so} \quad \operatorname{rank}(\bar{B}_m) \geq \operatorname{rank}(\bar{A}_m)$$

Same rank $\Leftrightarrow \det(\bar{B}_m) = 0$

$$\Leftrightarrow \quad 0 = -\frac{1}{2} m^{13}\left[6m^{-7} - 2m^{-5}\right]$$

Since $m > 0$, this is equivalent

to $\quad 6 = 2m^2, \quad m > 0$

$$\underline{\underline{m = \sqrt{3}}}$$

3 (a) For an eq. pt, $\dot{x} = \dot{y} = 0$

(i) $\dot{y} = 0 \iff G'(x) = m^3$ (as $y > 0$)

__at most__ one $\bar{x}$

since G' strictly monotone.

$\dot{x} = 0$ & $x = \bar{x}$

$\Updownarrow$

$H(y) = -G(\bar{x})$

__at most__ one $\bar{y}$ since H strictly monotone.

(ii) Jacobian $\bar{J} = \begin{pmatrix} G'(\bar{x}) & H'(\bar{y}) \\ -G''(\bar{x})\,\bar{y} & \underbrace{m^3 - G'(\bar{x})}_{=0} \end{pmatrix}$

$= \begin{pmatrix} m^3 & H'(\bar{y}) \\ -G''(\bar{x})\,\bar{y} & 0 \end{pmatrix}$

Saddle point because $\det(\bar{J})$

$= H'(\bar{y})\,G''(\bar{x})\,\bar{y}$ __< 0__

Addendum: the typewritten " ȳ" was missing.
Not detected in the process, because m=1 and thus ȳ=1 (except the "auxilliary" calculations).

3 cont'd,

$$G(x) = 2x^{1/2} \qquad H(y) = -2y^{-3/4}$$

$$(\bar{x}, \bar{y}) = (m^{-6}, m^4);$$

(b)
$$\bar{J} = \begin{pmatrix} m^3 & H'(m^4) \\ -G''(m^{-6})m^4 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} m^3 & \frac{3}{2} m^{4 \cdot \frac{-7}{4}} \\ \frac{1}{2}(m^{-6})^{-\frac{3}{2}} m^4 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} m^3 & \frac{3}{2} m^{-7} \\ \frac{1}{2} m^{9+4} & 0 \end{pmatrix}$$

Here was the error, see the grading guidelin

What this paper will do now:

- finish the calculations with general m
- point out that if m=1 the hint is "valid".

The simplest case m=1 first:

3(b) cont'd ; case $m=1$.

Then the Jacobian $\bar{J}$

equals $\bar{A}_1$ from 2(a),

The slope is the one of the

eigenvector corresponding to

the negative eigenvalue, i.e.

$$(-m^{-3}, m^7) = (-1, 1),$$

Slope $= \underline{\underline{-1}}$.

3 (b) Case m (possibly) $\neq 1$.

$$J = \begin{pmatrix} m^3 & \frac{3}{2} m^{-7} \\ \frac{1}{2} m^{13} & 0 \end{pmatrix}$$

Negative eigenvalue $\gamma = \frac{1}{2}\left[m^3 - \sqrt{m^6 + 3m^2} \right]$

$$= \frac{m}{2}\left[m^2 - \sqrt{m^4 + 3} \right]$$

Eigenvector satisfies

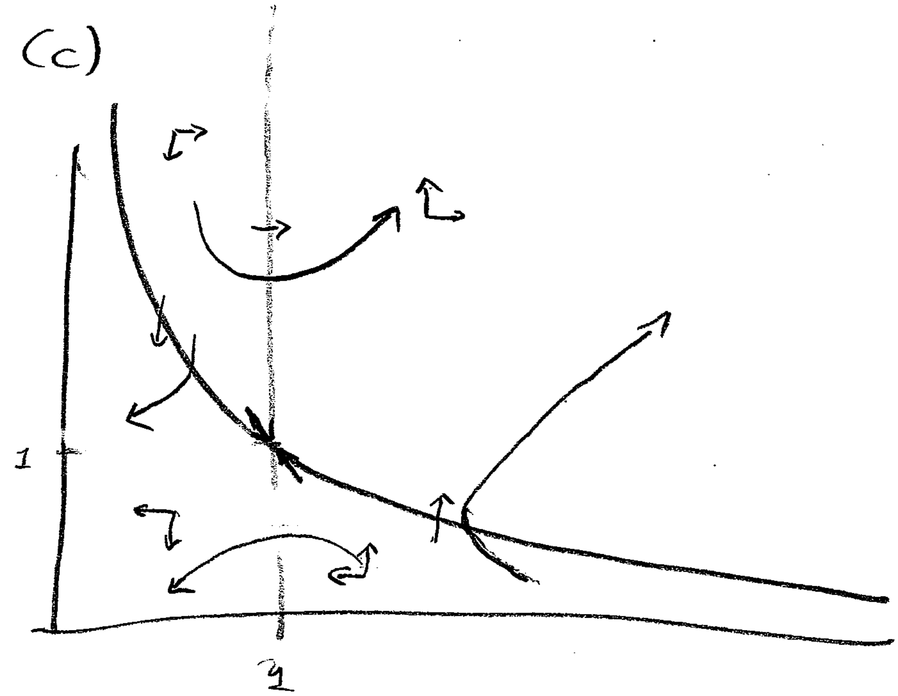
$$\left(\tfrac{1}{2} m^{13} \quad -\gamma \right) \vec{w} = 0$$

with slope $\dfrac{\frac{1}{2} m^{13}}{\gamma} = - \underline{\underline{\dfrac{m^{12}}{\sqrt{m^4+3} - m^2}}}$

(which, expanding the fraction with $m^2 + \sqrt{m^4+3}$, equals $\underline{-\frac{1}{3} m^{12}\left(m^2 + \sqrt{m^4+3}\right)}$)

(For $m = 1$ the answer of course reduces to -1.)

3 (c)



Problem 4

Let $H(t, x, u, p)$

$$= -6e^{-t}\frac{1}{u} + p\left(2x^{1/2} - 2u^3\right)$$

(a) conditions are:

u^* maximizes $u \mapsto H$ i.e., maximizes $-6e^{-t}\frac{1}{u} - 2pu^3$

$$\dot{p} = -\frac{\partial H}{\partial x} = -px^{-1/2}$$

with $p(2014) \geq 0$ $(= 0$ if $x(2014) > 0)$.

$$\dot{x}^* = 2x^{*1/2} - 2u^{*3} \quad \text{with } x(0) = x_0$$

4 (b)

$$\dot{y} = y + e^{t}\dot{p} = y - e^{t}p \cdot x^{-1/2}$$

$$= y \cdot [1 - x^{-1/2}]$$

$$= y \cdot [1 - G'(x)]$$

(as (D) with $m=1$)

For the maximum principle to hold, u^* must satisfy the 1st order condition

$$6e^{-t}u^{-2} = 6pu^{2}$$

$$u^{-4} = pe^{t} = y$$

$$u^* = y^{-1/4}$$

so $\dot{x} = G(x) - 2u^{*3}$

$= G(x) + H(y)$ as (D).

4(c)

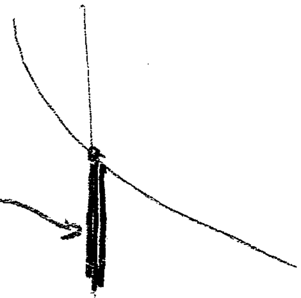
$$v'(1) = p(0) \quad (\text{for the case } x_0 = 1)$$

$$= y(0) \quad (\text{ — } \text{"} \text{ — })$$

In order for x to end up at 0, we must start **below** the saddle point path, i.e. $y(0) \le 1$,

i.e. $\underline{p(0) \le 1}$, when $x_0 = 1$.

($p(0) > 0$ for u^* to exist.)



Here

$x_0 = 1$