University of Oslo / Department of Economics / NCF

ECON4140 Mathematics 3 – on the 2015–05–29 exam

- This note is *not* suited as a complete solution or as a template for an exam paper, it is too sketchy. It was written as guidance for the grading process – however, with additional notes and remarks for using the document in teaching later.
- For readability, the problems are restated, their respective solutions on the same page.
- Weighting: assigned at the grading committee's discretion. (In case of appeals: the new grading committee assigns weighting at their discretion.) The problem set was written with the intention that a uniform weighting over letter-enumerated items should be a *feasible* choice, and this – along with it being merely an *intention to facilitate* which does not tie the committe's hands – has been communicated.

Problem 3 fits this page and the related problem 4 follows:

Problem 3 Let $0 < K < Q < 1$ be constants and let G be a given function. Consider the differential equation system

$$\begin{aligned} \dot{x}(t) &= p(t) + Q \\ \dot{p}(t) &= Kx(t) - G(t) \end{aligned} \tag{D}$$

(a) Deduce a second-order differential equation for x, and find the general solution of this equation when $G \equiv 0$. (*Hint:* For which γ will $x(t) = e^{\gamma t}$ be a particular solution?)

(b) Find the general solution of (D) for the case when $G(t) = Ke^t$.

On the solution of Problem 3

(a) We have $\ddot{x}(t) = \dot{p}(t)$, so the equation is $\ddot{x}(t) = Kx(t) - G(t)$. When $G = 0$ we have general solution $C_1 e^{t\sqrt{K}} + C_2 e^{-t\sqrt{K}}$ since $K > 0$.

(b) For a particular solution for x, try Le^t and fit L: $Le^t = KLe^t - Ke^t$, so that $L = K/(K-1)$. This gives x; then $p = \dot{x} - Q$:

$$\begin{aligned} x(t) &= C_1 e^{t\sqrt{K}} + C_2 e^{-t\sqrt{K}} + \frac{K}{K-1} e^t \\ p(t) &= \left(C_1 e^{t\sqrt{K}} - C_2 e^{-t\sqrt{K}}\right)\sqrt{K} + \frac{K}{K-1} e^t - Q \end{aligned}$$

Problem 4 Let $0 < K < Q < 1$ be constants, and consider the optimal control problem

$$\max_{u(t)\in\mathbf{R}} \int_0^{11} \Big\{ -\frac{K}{2}\cdot\big[x(t)-e^t\big]^2 - \frac{1}{2}\big[u(t)\big]^2\Big\}\,dt, \qquad \dot{x} = u + Q, \quad x(0) = x_0, \quad x(11) \text{ free.}$$

(a) i) State the conditions from the maximum principle.
ii) Are these conditions also sufficient?

(b) Show that in optimum, x and the adjoint (costate) p must satisfy the differential equation system (D) in problem 3, with $G(t) = Ke^t$.

(c) Suppose that for some set of parameters the optimal solution ends at $x(11) = 11e^{11}$. Approximately how much would the optimal *value* change if the final time were reduced from 11 to 10.9?

On the solution of Problem 4:

(a) Let $H(t, x, u, p) = -\frac{K}{2}(x - e^t)^2 - \frac{1}{2}u^2 + p(u + Q)$. For (x^*, u^*) to be optimal, there must be some $p = p(t)$ satisfying the following conditions:

- u^* maximizes H over $u \in \mathbb{R}$, i.e. maximizes $pu - \frac{1}{2}u^2$;
- $\dot{p}(t) = K(x^*(t) - e^t)$ with $p(11) = 0$
- $\dot{x}^* = u^* + Q$ with $x(0) = x_0$.

H is concave wrt. (x, u) (being a concave function wrt. x plus a concave wrt. u), so the conditions are sufficient.

(b) To satisfy the conditions, the optimal control is p, so that x satisfies (D); also, the equation for $\dot{p}$ is like in (D).

(c) The derivative wrt. final time is $H(11, x^*(11), u^*(11), p(11)) = -\frac{K}{2}(11e^{11}-e^{11})^2-0+0$, and a change of $-1/10$ yields a value change of $\approx \frac{K}{20}(10e^{11})^2 = 5Ke^{22}$.

Problem 1 Define for each $h \in \mathbf{R}$ the following matrices

$$\mathbf{A}_h = \begin{pmatrix} 5-h & 3 \\ 3 & 4-h \\ 2 & 3 \end{pmatrix}, \quad \mathbf{b}_h = \begin{pmatrix} 2 \\ 3 \\ 5-h \end{pmatrix}, \quad \mathbf{C}_h = \begin{pmatrix} 5-h & 3 & 2 \\ 3 & 4-h & 3 \\ 2 & 3 & 5-h \end{pmatrix}, \quad \mathbf{M} = \mathbf{C}_0$$

(where $\mathbf{C}_0$ denotes $\mathbf{C}_h$ with $h = 0$). Observe that $\mathbf{C}_h = \mathbf{M} - h\mathbf{I} = \big(\mathbf{A}_h|\mathbf{b}_h\big)$.

(a) $\mathbf{u} = (1, -2, 1)'$ is an eigenvector of $\mathbf{M}$. Find a corresponding eigenvalue λ_1. (You shall obtain that $0 < \lambda_1 < 3$.)

(b) $\lambda_2 = 3$ is an eigenvalue of $\mathbf{M}$. Find a corresponding eigenvector $\mathbf{v}$. (You shall obtain an answer such that $v_1 v_3 < 0$.)

(c) It is a fact that $\mathbf{M}$ has an eigenvector $\mathbf{w}$ with all coordinates nonnegative. Show why this fact together with parts (a) and (b) imply that $\mathbf{M}$ must be positive definite. (You are required to use precisely these pieces of information; you will not be rewarded for using other calculations.)

(d) Show that $\mathbf{A}_h$ has rank 2 no matter what h is.

(e) Decide whether the following statement is true or false: *"The equation system $\mathbf{A}_h\binom{p}{q} = \mathbf{b}_h$ has a solution $\binom{p}{q}$ if and only if h is an eigenvalue for $\mathbf{M}$."*

On the solution of Problem 1

(a) Calculate $\mathbf{Mu}$ to get $\mathbf{u}$, so that $\lambda_1 = 1$.

(b) The first and last row of $\mathbf{C}_3$ are the same (delete one), while the top-left 2×2 minor is nonzero. Subtract $3/2$ of the first row from the second to get that $v_2 = 0$. Then $v_1 + v_3 = 0$, so $\mathbf{v} = (1, 0, -1)'$ (or any nonzero scaling) is an eigenvector corresponding to $\lambda_2 = 3$.

(c) From parts (a) and (b), $\mathbf{w}$ is indeed a *third* eigenvector, and since λ_1 and λ_2 are > 0, we have $\mathbf{M}$ positive definite iff the third eigenvalue is positive too. Which it is: Because each element of $\mathbf{Mw}$ is the sum of nonnegative numbers – not all zero, because $\mathbf{M}$ isn't null and $\mathbf{w}$ is an eigenvector and cannot be null – the eigenvalue cannot be ≤ 0.

(d) The bottom 2×2 minor is nonzero except when $8 - 2h = 9$ i.e. $h = -1/2$. For $h = -1/2$, some other 2×2 minor is nonzero: $\left|\begin{smallmatrix} 5-h & 3 \\ 2 & 3 \end{smallmatrix}\right| = 9 - 3h$ is > 0 for $h = -1/2$ (and the last 2×2 minor is nonzero too).

(e) True: there is solution iff $\mathbf{C}_h$ and $\mathbf{A}_h$ have same rank, and since $\mathbf{A}_h$ is a block in $\mathbf{C}_h$, then $\text{rank}(\mathbf{C}_h) \geq \text{rank}(\mathbf{A}_h) = 2$. Thus the ranks match iff $\text{rank}(\mathbf{C}_h) < 3$ i.e. iff $0 = |\mathbf{C}_h| = |\mathbf{M} - h\mathbf{I}|$ i.e. iff h is an eigenvalue for $\mathbf{M}$.

Problem 2 Given constants $r \geq 0$, $s > 0$ and $t > 0$, a vector $\mathbf{m} \in \mathbf{R}^n$ such that $1 = m_1 \geq m_2 \geq \ldots m_n \geq 0$, and for $\mathbf{x} \in \mathbf{R}^n$ the functions

$$g(\mathbf{x}) = |x_1| + \ldots + |x_n|, \qquad F(\mathbf{x}) = \mathbf{m}'\mathbf{x} - sg(\mathbf{x}) + (s-1)t, \qquad H(\mathbf{x}) = F(\mathbf{x}) - r \max_i |x_i|$$

(where $\max_i |x_i|$ means the greatest of the n numbers $|x_1|, \ldots, |x_n|$).

(a) i) Show that H is concave for every $r \geq 0$, $s > 0$.
ii) Consider part (b) below. Explain why the existence of such an s as asked for in part (b), will show that $\mathbf{x}^* = (t, 0, \ldots, 0)'$ solves the nonlinear programming problem

$$\max_{\mathbf{x}} \ \mathbf{m}'\mathbf{x} \quad \text{subject to} \quad g(\mathbf{x}) \leq t$$

(b) Find an $s \in [0, 1]$ such that $\mathbf{0}$ is a supergradient for F at $\mathbf{x}^* = (t, 0, \ldots, 0)'$.
Hint: Explain why it suffices to show that F attains a (local or global) maximum at $\mathbf{x}^*$, and then show that this happens for some $s \geq 0$. You shall get that $m_n \leq s \leq m_1$ and also that s does not depend on t (if you need to, check the case $t = 1$ first).

On the solution of Problem 2

(a) i): the absolute value is a convex function, the max of convexes is convex, and $-r \leq 0$. The linear and constant terms do not affect concavity/convexity, and since $s > 0$ it suffices to show g convex – and it is a sum of convexes.
ii): F is the Lagrangian(*) of the problem, with s being the multiplier. Part (b) then restates the sufficient condition for the concave programming problem to have a solution at $\mathbf{x}^*$ (where the constraint is active, so any $s \geq 0$ will do – we need not have $s \leq 1$, but it is certainly sufficient).

(b) For a local max, $\mathbf{0}$ is a supergradient. We have $F(\mathbf{x}) = \sum_i \big(m_i x_i - s|x_i|\big)$ plus a constant, and it suffices to find an s such that $\mathbf{x}^*$ (locally) maximizes. We can consider each coordinate x_i separately: Any $s \geq m_2$ will make 0 maximize $m_i z - s|z|$ for $i \geq 2$, while $s = m_1 = 1$ makes $m_1 z - s|z|$ identically zero for $z \geq 0$ – hence $z = t$ is a local max.

(*) Note added 2016: The Lagrangian is actually $F+t$, where t is a constant and does not change any conditions.