University of Oslo / Department of Economics **(English only)**

ECON4140 Mathematics 3

June 10th 2016, 0900–1200.

There are 2 pages of problems to be solved.

All printed and written material may be used, as well as pocket calculators.

Grades given run from A (best) to E for passes, and F for fail.

- You are required to state reasons for all your answers.
- You are permitted to use any information stated in an earlier letter-enumerated item (e.g. "(a)") to solve a later one (e.g. "(c)"), regardless of whether you managed to answer the former. A later item does not necessarily require answers from or information given in a previous one.

Suggested weighting: Problems 1, 2, 3 and 4 are *expected* to count 25 percent each.

Problem 1 For arbitrary real constants a, b, c, d, p, q, r, s, define the matrices

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}, \qquad \mathbf{F} = \begin{pmatrix} a & 0 & 0 & b \\ 0 & p & q & 0 \\ 0 & r & s & 0 \\ c & 0 & 0 & d \end{pmatrix}, \qquad \mathbf{G} = \begin{pmatrix} a & 0 & 0 & b & a & b \\ 0 & p & q & 0 & c & d \\ 0 & r & s & 0 & a & b \\ c & 0 & 0 & d & c & d \end{pmatrix}$$

Notice that $\mathbf{F}$ has $\mathbf{B}$ as the "middle" block and the elements of $\mathbf{A}$ in the corners, and that the blocks of $\mathbf{G}$ are $\left(\mathbf{F} \vdots \begin{smallmatrix}\mathbf{A} \\ \mathbf{A}\end{smallmatrix}\right)$. Let prime (e.g. $\mathbf{F}'$) denote transpose.

(a)
- Show that if $qr = 0$, then either $(0, 1, 0, 0)'$ or $(0, 0, 1, 0)'$ is an eigenvector of $\mathbf{F}$, and
- find the eigenvalue corresponding to either $(0, 1, 0, 0)'$ or $(0, 0, 1, 0)'$.

(b) If $qr = ps \neq 0$, then $p + s$ is an eigenvalue of $\mathbf{F}$.
- Find a corresponding eigenvector of the form $(h, k, \ell, 0)'$.

(c) Show that for λ to be an eigenvalue of $\mathbf{F}$, then λ must be an eigenvalue of $\mathbf{A}$ or an eigenvalue of $\mathbf{B}$. (Or possibly both. *Hint:* what is the product of the characteristic polynomials of $\mathbf{A}$ and $\mathbf{B}$?)

Problem 2

(a) Solve the difference equation

$$x_{t+2} + \frac{1}{6}x_{t+1} - \frac{1}{6}x_t = 2 \tag{D}$$

(b)
- Show that the differential equation system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 6 \end{pmatrix} \tag{S}$$

  implies that x satisfies

$$\ddot{x} + \frac{1}{6}\dot{x} - \frac{1}{6}x = 2 \tag{E}$$

- Find the general solution of (E)

(c)
- Is (D) stable?
- Is (E) stable?
- Classify the equilibrium point $(x^*, y^*) = (-12, 6)$ of (S).

Problem 3 Let $a_t > 0$, $b_t > 0$, $m > 1$, $T > t_0$ and $x_{t_0} > 0$ be given numbers. Consider the dynamic programming problem

$$J_{t_0}(x_{t_0}) = \max_{u_t \in [0,1]} \sum_{t=t_0}^{T} a_t^{1/m} \cdot \big([u_t + b_t] \cdot x_t\big)^{1-1/m}, \qquad \text{subject to} \quad x_{t+1} = x_t \cdot (1 - u_t)$$

Although you cannot use without proof the information given in part (b) in part (a), it gives a hint that is likely helpful.

(a) In this part, let $a_t = b_t = 1$ for all t. Calculate J_T, J_{T-1} and J_{T-2}.

(b) Show by induction that the optimal u_t^* does not depend on x.

Problem 4 Let r, k, T and x_0 be given numbers, all > 0, and let n be a natural number ($\in \{1, 2, 3, \dots\}$). Consider the optimal control problem

$$\max_{u(t) \geq 0} \int_0^T \Big[kx(t) + e^{-rt}\big(u(t)\big)^{1-1/2n}\Big]\,dt, \qquad \text{subject to} \quad \dot{x} = -u, \qquad x(0) = x_0, \quad x(T) \geq 0.$$

(a)
- State the conditions from the maximum principle. (Put the «p_0» constant $= 1$.)
- Do we know that a pair (x^*, u^*) satisfying these conditions, will be optimal?

(b) Show that there are constants α, β and γ such that
- the optimal control is of the form $u^*(t) = \big(\alpha + \beta \cdot (t - T)\big)^{-2n} e^{-\gamma t}$, with
- $\alpha \geq 0$.