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2017-05-16 (NCF):
Corrected sign in
4(b) and some typos.
Corrections appear
in this font.
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ECON4140, the 2016 exam

No formal guidelines were produced prior to grading. This note summarizes the experiences from the process.

The suggested weighting of 1/4 on each of problems 1, 2, 3 and 4, was carried out. By and large, parts 1 (a)–(c) and 2 (a)–(c) were given equal weight, and parts 3(a), 3(b), 4(a) and 4(b) were given equal weight. In some cases, particularly in problem 3, some papers were far into part (b) under a "3 (a)" headline, and got partial credit for part (b) as well.

Some considerations for the benefit of (most of) the candidates, had to be made on problems 3 and 4, where lots of elementary errors (which were not really core to the theory) could very well have been penalized *much* harsher. Nevertheless, both problems 1, 3 and 4 ended up on an average score of 39 percent, just below the pass mark at 40 percent. The reason why the grading distribution in the end turned out to look fairly normal, was the good answers to problem 2, where the average score was worth a B. The usual grading thresholds of 40, 45, 55, 75 and 91 percent were applied.

Problems 1 through 4 are addressed on the next four pages.

Problem 1

Matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{F}$ were also assigned in Mathematics 2. The impression – without any rigorous count being made – is that the Mathematics 3 candidates messed up elementary calculations (like cofactor expansion in part (c)) more than the Mathematics 2 students did. The matrix $\mathbf{G}$ was not used for anything; the explanation was that at the final stage, one question was dropped in the interest of reducing total workload.

Part (a) Should be straightforward, but got only about 1/2 score. Direct calculations:

$$\mathbf{F}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}=\begin{pmatrix}0\\p\\r\\0\end{pmatrix}\qquad\text{and}\qquad\mathbf{F}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}=\begin{pmatrix}0\\q\\s\\0\end{pmatrix}$$

By assumption, $q=0$ or $r=0$. If $r=0$, the former is a scaling $\underline{\underline{p}}$ of $(0,1,0,0)'$ which is then an eigenvector as it should; if $s=0$, the latter will be a scaling $\underline{\underline{s}}$ of $(0,0,1,0)'$ which is then an eigenvector.

Problem (b) This got about 1/3 average score. Solve $(\mathbf{F}-(p+s)\mathbf{I})\,\mathbf{x}=\mathbf{0}$ using Gaussian elimination. Except for special values of a and c, $x_1=h$ must vanish. We are left with solving $\left(\begin{smallmatrix}-s & q\\ r & -p\end{smallmatrix}\right)\left(\begin{smallmatrix}k\\ \ell\end{smallmatrix}\right)=\mathbf{0}$. Putting $\underline{\underline{k=q \text{ and } \ell=s}}$ will work in the first equation (this is not a null vector, as $qr=ps\neq 0$) and thus automatically in the second, as there is some degree of freedom given that it is known (from the problem text) that we have an eigenvalue. Any nonzero scaling (like $\left(\begin{smallmatrix}1\\ s/q\end{smallmatrix}\right)$) will of course also be a correct answer. Also $\left(\begin{smallmatrix}p\\ r\end{smallmatrix}\right)$ (or any nonzero scaling), using the second equation instead.

Part (c) Average score even worse than part (b), despite not being a hard problem; the Mathematics 2 exam asked for $|\mathbf{F}|$, and the calculations of $|\mathbf{F}-\lambda\mathbf{I}|$ is analogous with the same zero elements and two steps of cofactor expansion. Along the first row:

$$|\mathbf{F}-\lambda\mathbf{I}|=(a-\lambda)\begin{vmatrix}p-\lambda & q & 0\\ r & s-\lambda & 0\\ 0 & 0 & d-\lambda\end{vmatrix}-b\begin{vmatrix}0 & p-\lambda & q\\ 0 & r & s-\lambda\\ c & 0 & 0\end{vmatrix}$$

By a second cofactor expansion along the last row: $(a-\lambda)(d-\lambda)\left|\begin{smallmatrix}p-\lambda & q\\ r & s-\lambda\end{smallmatrix}\right|-bc\left|\begin{smallmatrix}p-\lambda & q\\ r & s-\lambda.\end{smallmatrix}\right|$. The common factor is $|\mathbf{B}-\lambda\mathbf{I}|$, and factoring out, we have $|\mathbf{A}-\lambda\mathbf{I}|\cdot|\mathbf{B}-\lambda\mathbf{I}|$, the product of characteristic polynomials. An eigenvalue of $\mathbf{F}$ is a zero of $|\mathbf{F}-\lambda\mathbf{I}|$ is a zero of $|\mathbf{A}-\lambda\mathbf{I}|\cdot|\mathbf{B}-\lambda\mathbf{I}|$ is a zero of $|\mathbf{A}-\lambda\mathbf{I}|$ or of $|\mathbf{B}-\lambda\mathbf{I}|$ (or both) – is an eigenvalue of $\mathbf{A}$ or of $\mathbf{B}$ (or of both).

Some papers messed up the cofactor expansion. Other papers computed the characteristic polynomials and claimed the identity hinted at (whether or not they even obtained it, given their errors!) and then stopped without pointing out any connection between $|\mathbf{F}-\lambda\mathbf{I}|$ and eigenvalues; this connection cannot be taken as "goes-without-saying" for candidates who in parts (a) and (b) could hardly show any knowledge of eigenvalues and their properties.

Problem 2

Problem 2 turned out with high scores, especially part (b), where the average was closing in on the "A". The problems are not so hard, but there are a couple of pitfalls that led to somewhat weaker answers to the difference equations questions.

The deduction of (E) from (S) by differentiating the first equation in the system, turned out fine in the sense of score – although it is easy to make mistakes that take time to debug, the answer is given.

The coefficients of (D) and (E) are the same, and so are the characteristic roots, solving $r^2 + r/6 - 1/6 = 0$, i.e., roots $r_1 = -1/2$ and $r_2 = 1/3$. The corresponding homogeneous equations will then have solutions $A_1(-1/2)^t + A_2(1/3)^t$ and $C_1 e^{-t/2} + C_2 e^{t/3}$, respectively. Now, the stability criteria are *different,* a pitfall that did catch a few candidates (most of whom applied the differen*tial* equations criterion). The roots are real, so we need not invoke the "real part" terminology (and candidates were free to phrase the criteria as if roots were always real): (D) is `stable` (even, globally asymptotically stable) because $\max\{|r_1|, |r_2|\} < 1$, while (E) is unstable as $\max\{r_1, r_2\} > 0$; because $r_1 < 0$, the equilibrium point of (S) is a saddle point. Stability and instability could easily be inferred from the solutions as well – even the saddle point classification could, as the convergent paths are found by putting $C_2 = 0$.

Now for the (constant!) particular solution of the inhomogeneous (D) and (E), they are $\underline{\underline{2}}$ resp. $\underline{\underline{-12}}$, hence *not* the same – again, most who erred, picked the formula for the differen*tial* equations, this explaining the nearly one grade difference between parts (a) and (b).

Problem 3

This was less than satisfying. We think even ECON2200 candidates should know that root functions are monotonous, and that $z^{\text{negative number}}$ does *not* vanish. Nevertheless, several candidates computed J_T by differentiating, getting $(1+u)^{-1/m} = 0$, inferring that $1+u=0$ and then either putting $u_T^* = -1$ without a second thought, or pointing out that we are restricted to $[0,1]$ and thus picking the closest point 0 (which, although wrong, would be a valid inference had one pointed out concavity, which one should be able to at this stage). We recognize though, that economics candidates would be inclined to look for stationary points unless drilled at other options, and that the $-1/m$ formulation could invite to inverting.

We ended up with awarding a middle "B" on part (a) for carrying out a dynamic programming argument, even when the static optimization was done wrong[1]. On part (b), a middle "B" was awarded for carrying out the induction proof regardless of whether one kept $a_t \equiv b_t \equiv 1$ (as many did), or messed up the static optimization.

These measures were applied under considerable doubt. Yet the average score on problem 3 still ended up (just) below the pass mark, with part (b) being the worst in the set – despite its similarity to a problem known from the seminars.

Part (a) Let $a_t \equiv b_t \equiv 1$. We have $J_T(x) = \max_{u\in[0,1]} \{((1+u)x)^{1-1/m}\}$; the maximand is strictly increasing in u, so $\underline{\underline{J_T(x) = (2x)^{1-1/m} \text{ with } u_T^* = 1}}$.

$J_{T-1}(x) = \max_{u\in[0,1]} \{((1+u)x)^{1-1/m} + (2x(1-u))^{1-1/m}\} = x^{1-1/m}\max_{u\in[0,1]} g_{T-1}(u)$, where $g_{T-1}(u) = (1+u)^{1-1/m} + 2^{1-1/m}(1-u)^{1-1/m}$ has derivative $[1-1/m]\cdot[(1+u)^{-1/m} - 2^{1-1/m}(1-u)^{-1/m}]$ and is concave (differentate once more and notice that $[1-1/m]\cdot[-1/m] < 0$). The derivative at zero is negative, so $\underline{\underline{u_{T-1}^* = 0}}$ and $J_{T-1}(x) = x^{1-1/m} g_{T-1}(0) = \underline{\underline{(1+2^{1-1/m})x^{1-1/m}}}$.

$J_{T-2}(x) = \max_{u\in[0,1]} \{((1+u)x)^{1-1/m} + (1+2^{1-1/m})x^{1-1/m}(1-u))^{1-1/m}\}$. We maximize $g_{T-2}(u) = (1+u)^{1-1/m} + (1+2^{1-1/m})(1-u)^{1-1/m}$ (concave again, its derivative at zero is (even more!) negative). So $\underline{\underline{u_{T-2}^* = 0}}$ and $J_{T-2}(x)$ $\underline{\underline{(2+2^{1-1/m})x^{1-1/m}}}$.

Part (b) With general a_t and b_t, a positive $x^{1-1/m}$ factors out: suppose for induction that $J_t(x) = G_t x^{1-1/m}$ with u_t^* not depending on x. Then at time $t-1$:

$$\begin{aligned} J_{t-1}(x) &= \max_{u\in[0,1]} \left\{ a_t^{1/m} \cdot \big([u_t + b_t]\cdot x_t\big)^{1-1/m} + G_t \cdot (x(1-u))^{1-1/m} \right\} \\ &= x_t^{1-1/m} \underbrace{\max_{u\in[0,1]} \left\{ a_t^{1/m} \cdot [u_t + b_t]^{1-1/m} + G_t \cdot (1-u)^{1-1/m} \right\}}_{=:G_{t-1},\text{ no }x\text{-dependence}} \end{aligned}$$

so neither will u_{t-1}^* nor G_{t-1} depend on x. The induction proof is completed by noting that when $t = T$, the property holds: $G_T = \max_{u\in[0,1]} \{a_T^{1/m} \cdot [u_T + b_T]^{1-1/m}\}$, the maximizing u_T^* does not depend on x as the maximization does not.

[1] as long that it did not over-simplify; the candidate who claimed that we needed $x = 0$ and hence $J_t = 0$ identically, could certainly not get such a score

Problem 4

Again, there were some strange recurring issues. About half of the papers seem to think that the maximum principle states `H'`$_u = 0$ (failing to capture non-stationary boundary optima). Part (a) ended up being scored roughly as follows: 20 percent for getting the Hamiltonian, equally much for the maximization (half deducted for claiming stationarity is necessary), 30 percent for p with transversality and 30 percent for pointing out that the concavity makes the conditions sufficient as well. The average score for part (a) was around 1/2.

We chose to be forgiving over the mistake of `identifying` $p(0)$ with the "p_0" constant (which the book uses in front of the running utility as a Fritz John type formulation capturing the constraint qualification), and the consequences. Hardly anyone got full score on part (b), and the average was below the pass mark threshold.

Part (a) With $H(t, x, u, p) = kx + e^{-rt}u^{1-1/2n} - pu$, we did expect the following conditions:

- That the optimal control u^* maximizes $kx + e^{-rt}u^{1-1/2n} - pu$ over $u \geq 0$;
- That p satisfies $\dot{p} = -\partial H/\partial x = -k$ with $p(T) \geq 0$ and $p(T) > 0$ should $x^*(T) > 0$.

We do not stress that the optimal path must satisfy the differential equation, as that could, arguably, be a "goes-without-saying".

For sufficiency, we got some strange arguments confusing single-variable and two-variable concavity, that should be well known. However, most who tried to answer this, relied on trying to verify the Mangasarian condition: we have a sum of concave functions, as $u^{1-1/2n}$ is concave. Alternatively, one could point out that as the maximization wrt. u does not involve x, then the maximized Hamiltonian will be kx (linear, thus concave) plus something not involving x, verifying the Arrow condition.

Part (b) Here, we would have wanted to see the following: H'_u is infinite at $u = 0$, so a maximum must be interior (this detail we hardly saw); the first-order condition yields

$$(1 - 1/2n)e^{-rt}u^{-1/2n} = p$$

which when solved out, and inserting for $p(t) = p(T) + k(T - t)$ from the differential equation, yields

$$u^*(t) = \left(\frac{p(t)}{1 - 1/2n}\right)^{-2n} e^{-2nrt} = \left(\frac{p(T) + k(T - t)}{1 - 1/2n}\right)^{-2n} e^{-2nrt}$$

We can now identify the constants, even on closed form:

$$\underline{\underline{\alpha = \frac{p(T)}{1 - 1/2n}, \quad \beta = \frac{-\,k}{1 - 1/2n} \quad \text{and} \quad \gamma = 2nr.}}$$

Corrected the sign of beta.

For the final question, we would have wanted to see that $\alpha \geq 0$ because the transversality condition yields $p(T) \geq 0$.

ECON4140 exam 2016: comments to problem 4, one page