ECON4140 Mathematics 3

June 1st 2018, 1430–1730.

There are 2 pages of problems to be solved.

All printed and written material may be used, as well as both the approved calculators.

Grades given run from A (best) to E for passes, and F for fail.

- You are required to state reasons for all your answers.
- You are permitted to use any information stated in an earlier letter-enumerated item (e.g. "(a)") to solve a later one (e.g. "(c)"), regardless of whether you managed to answer the former. A later item does not necessarily require answers from or information given in a previous one.

Problem 1 Let $\mathbf{A}_c = \begin{pmatrix} 0 & 0 & 9 \\ 0 & c & 0 \\ 5 & 0 & 4 \end{pmatrix}$ for each real constant c.

(a) Decide the rank of $\mathbf{A}_c$ and the definiteness of the quadratic form $Q(\mathbf{x}) = \mathbf{x}'\mathbf{A}_c\mathbf{x}$.

(b) $\mathbf{v} = (9, 0, -5)'$ is an eigenvector. Find the associated eigenvalue $\nu < 0$.

(c) Calculate the characteristic polynomial $p(\lambda)$ and show that $p(c) = 0$.

(d) Find an eigenvalue $\mu > \nu$ such that μ does not depend on c, and an associated eigenvector $\mathbf{u}$.

Problem 2 Consider the difference equation $x_{t+2} - x_{t+1} + x_t = \kappa 2^t$.

(a) In this part, let $\kappa = 0$. Find the particular solution that satisfies $x_0 = 0$ and $x_1 = 1$.

(b) In this part, let $\kappa = 1$. Find the general solution.

(c) Let $a_n = \dfrac{1}{n! \cdot (n+2)}$. Prove by induction that $a_1 + \ldots + a_n = \dfrac{1}{2} - \dfrac{1}{(n+2)!}$.

Problem 3 Let $a \geq b$ be constants, either both > 1 or both $\in (-1, 1)$ (i.e., either $a \geq b > 1$ or $1 > a \geq b > -1$). Consider the differential equation system

$$\begin{aligned} \dot{x} &= (1 - ay - x) \cdot x \\ \dot{y} &= (1 - bx - y) \cdot y \end{aligned} \tag{S}$$

(a)
- Find all four stationary states (equilibrium points).
- Classify that stationary state $(\bar{x}, \bar{y})$ for which both $\bar{x} > 0$ and $\bar{y} > 0$. (Such a point does exists under the assumptions on the constants. Your answer might depend on a and b.)

(b) Let $a = b = \frac{1}{2}$. Sketch a phase diagram covering the set where $0 \leq x \leq 2$, $0 \leq y \leq 2$, and indicate some solution curves.

Problem 4 Consider for constants $x_0 > 0$ and $T > 0$ the optimal control problem

$$\max_{u(t) \in [0,1]} \int_0^T (u - x^2)\, dt \qquad \text{where } \dot{x} = x + u, \quad x(0) = x_0 \quad \text{and } x(T) \text{ free.}$$

(a) State the conditions from the maximum principle. Are these conditions also sufficient?

(b) Show that an optimal control u^* must be 0 or 1 somewhere in the open interval $(0, T)$. (I.e., that it *cannot* be optimal to choose a u s.t. $u(t) \in (0, 1)$ for all $t \in (0, T)$.)

Problem 5 Define the functions u and v on the (convex!) set $\{(x, y);\ x \geq 0,\ y \geq 0\}$ by

$$u(x, y) = (16xy)^3 \qquad \text{and} \qquad v(x, y) = x + \sqrt{x^2 + 2y}$$

(a) Decide quasiconcavity/quasiconvexity of each of the functions u and v.
Hints: (I) "neither" is wrong answer! (II) solve level curves $v(x, y) = C$ for y.

Consider now the *necessary* Kuhn-Tucker conditions – *disregarding* constraint qualifications, which you can take for granted that hold – associated to each of the problems

$$\begin{aligned} &\max\ u(x, y) && \text{such that} && v(x, y) = 1, \quad x \geq 0, \quad y \geq 0 && \text{(P1)} \\ &\max\ (-v(x, y)) && \text{such that} && u(x, y) \geq 1, \quad x \geq 0, \quad y \geq 0 && \text{(P2)} \end{aligned}$$

It is a fact that $(x_1, y_1) = (\frac{1}{2}, 0)$ satisfies the necessary conditions associated to (P1), and that $(x_2, y_2) = (\frac{1}{4}, \frac{1}{4})$ satisfies the necessary conditions associated to (P2).

(b) For each problem (P1), resp. (P2), and the corresponding point (x_1, y_1), resp. (x_2, y_2):

Does the point (x_1, y_1) resp. (x_2, y_2) also satisfy *sufficient* Kuhn–Tucker conditions? If not: which part of the conditions fails?

Hint: The Lagrangians are *not* concave.