University of Oslo / Department of Economics / NCF

ECON4140 Mathematics 3 – on the 2018–06–01 exam

- New this semester: restricting calculators to the scientific calculator Casio FX-85EX (as well as a simpler arithmetic one).
- *Standard disclaimer:* This note is not suited as a complete solution or as a template for an exam paper. It was written as guidance for the grading process – however, with additional notes and remarks for using the document in teaching later.
  - The document reflects what was expected in that particular semester, and which may not be applicable to future semesters. In particular, what tests one is required to perform before answering «no conclusion» may not apply for later.
- Weighting: at the committee's (and in case of appeals: the new grading committee's) discretion. The problem set was written with the intention that a uniform weighting over letter-enumerated items should be a *feasible* choice.
- Default percent score to grade conversion table for this course:

F (fail)	E	D	C	B	A
0 to 39	40 to 44	45 to 54	55 to 74	75 to 90	91 to 100

  The committee (and in case of appeals, the new committee) is free to deviate.

  Addendum after grading: See the attached note from the grading committee.

Problems restated as given, followed by annotations boxed. *Addendum after grading:* A two-page note from the grading committee is attached at the end.

Problem 1 Let $\mathbf{A}_c = \begin{pmatrix} 0 & 0 & 9 \\ 0 & c & 0 \\ 5 & 0 & 4 \end{pmatrix}$ for each real constant c.

(a) Decide the rank of $\mathbf{A}_c$ and the definiteness of the quadratic form $Q(\mathbf{x}) = \mathbf{x}'\mathbf{A}_c\mathbf{x}$.

(b) $\mathbf{v} = (9, 0, -5)'$ is an eigenvector. Find the associated eigenvalue $\nu < 0$.

(c) Calculate the characteristic polynomial $p(\lambda)$ and show that $p(c) = 0$.

(d) Find an eigenvalue $\mu > \nu$ such that μ does not depend on c, and an associated eigenvector $\mathbf{u}$.

On problem 1

(a) $\texttt{det}\,(\mathbf{A}_c) = 9 \cdot (-c \cdot 5)$ is nonzero (and the rank is 3) if and only if $c \neq 0$. The first and third rows are non-proportional and therefore linearly independent. So $\texttt{rank}\ \mathbf{A}_c = 3$ for $c \neq 0$, and $\texttt{rank}\ \mathbf{A}_0 = 2$.

For the quadratic form: writing out the function $Q(x, y, z) = cy^2 + 14xz + z^2$, we can attain any sign by fixing $z \neq 0$ and y and letting x vary. So Q is indefinite. If one alternatively uses matrix tools, the associated *symmetric* matrix is $\frac{1}{2}(\mathbf{A}_c + \mathbf{A}_c') = \begin{pmatrix} 0 & 0 & 7 \\ 0 & c & 0 \\ 7 & 0 & 4 \end{pmatrix}$. Indefiniteness follows from negativity of the minor $\texttt{det}\begin{vmatrix} 0 & 7 \\ 7 & 4 \end{vmatrix}$.

(Note on symmetrization: They have been urged to symmetrize in the course, and it was reiterated in the final review. Still, for those who *really* know what they are doing, it *is* possible to argue for *indefiniteness* based on opposite-sign eigenvalues if they are found first (then $Q(\mathbf{u}) = \mu\|\mathbf{u}\|^2 = 9\|\mathbf{u}\|^2 > 0 > \nu\|\mathbf{v}|^2 = Q(\mathbf{v})$) – but the course has not highlighted those one-sided implications.)

(b) $\mathbf{A}_c\mathbf{v} = (-45, 0, 45 - 20)' = -5\mathbf{v}$, so $\nu = -5$.

(c) $p(\lambda) = \texttt{det}\,(\mathbf{A}_c - \lambda\mathbf{I}) = \begin{vmatrix} -\lambda & 0 & 9 \\ 0 & c-\lambda & 0 \\ 5 & 0 & 4-\lambda \end{vmatrix}$. Already here we spot that $p(c) = 0$. Cofactor expansion along the middle column yields $(c - \lambda)(\lambda^2 - 4\lambda - 45)$.

(d) The trace $4 + c$ equals the sum $-5 + c + \mu$, so $\mu = 9$. (Alternatively, find the second zero of $\lambda^2 - 4\lambda - 45$ as $2 + \sqrt{4 + 45} = 9$.) Solving $(\mathbf{A}_c - 9\mathbf{I})\mathbf{u} = \mathbf{0}$ yields $u_1 = u_3$, and $u_2 = 0$ (not uniquely so if $c = 9$, but always possible). So an associated eigenvector $\mathbf{u}$ is $(1, 0, 1)'$.

(It was deliberate to ask for "an" associated eigenvector, so it is not necessary to cover all nonzero scalings.)

2019: In (d), before "I", inserted a missing "9" (typewriter-font).

Problem 2 Consider the difference equation $x_{t+2} - x_{t+1} + x_t = \kappa 2^t$.

(a) In this part, let $\kappa = 0$. Find the particular solution that satisfies $x_0 = 0$ and $x_1 = 1$.

(b) In this part, let $\kappa = 1$. Find the general solution.

(c) Let $a_n = \dfrac{1}{n! \cdot (n+2)}$. Prove by induction that $a_1 + \ldots + a_n = \dfrac{1}{2} - \dfrac{1}{(n+2)!}$.

On problem 2

(a) Homogeneous difference equation. The characteristic equation associated to the difference equation is $m^2 - m + 1 = 0$, with non-real roots $\frac{1}{2}(1 \pm \sqrt{-3})$ and trigonometric solutions $1^t[C\cos(t\pi/3) + D\sin(t\pi/3)]$. $x_0 = 0$ yields $C = 0$ and then $x_1 = 1$ yields $D = 1/\sin(\pi/3)$ (which actually equals $\sqrt{4/3} = \frac{2}{3}\sqrt{3}$, but that is not necessary to point out). Solution: $\underline{\underline{\sin(t\pi/3)/\sin(\pi/3)}}$.

(Note: Solving the first few by hand, we get $0, 1, 1, 0, -1, -1, 0, 1, \ldots$ so one can see without the formula that (i) we have period six, so the argument of the sin should be $t \cdot 2\pi/6$, and (ii) the amplitude does not change, hence the «1^t».)

(b) We already have the form for the corresponding homogeneous equation, and need some u_t^* satisfying the inhomogeneous. Try $K \cdot 2^t$ and fit K to $K[4 \cdot 2^t - 2 \cdot 2^t + 2^t] = 2^t$. This yields $K = 1/3$, so the answer is $\underline{\underline{A\cos(t\pi/3) + B\sin(t\pi/3) + \frac{1}{3} \cdot 2^t}}$.

(c) (High school exam R2 spring 2017, slightly modified.)
True for $n = 1$: $\frac{1}{2} - \frac{1}{(n+2)!}\Big|_{n=1} = \frac{1}{2} - \frac{1}{6} = \frac{1}{3} = a_1$, so suppose true for N. Then

$$a_1 + \ldots + a_{N+1} = a_1 + \ldots + a_N + a_{N+1} = \frac{1}{2} - \frac{1}{(N+2)!} + \frac{1}{(N+1)!(N+3)}$$
$$= \frac{1}{2} - \left[\frac{N+3}{(N+3)!} - \frac{N+2}{(N+3)!}\right] = \frac{1}{2} - \frac{1}{(N+3)!}$$

i.e., true for $N + 1$. We are done.

Problem 3 Let $a \geq b$ be constants, either both > 1 or both $\in (-1, 1)$ (i.e., either $a \geq b > 1$ or $1 > a \geq b > -1$). Consider the differential equation system

$$\begin{aligned} \dot{x} &= (1 - ay - x) \cdot x \\ \dot{y} &= (1 - bx - y) \cdot y \end{aligned} \tag{S}$$

(a)
- Find all four stationary states (equilibrium points).
- Classify that stationary state $(\bar{x}, \bar{y})$ for which both $\bar{x} > 0$ and $\bar{y} > 0$. (Such a point does exists under the assumptions on the constants. Your answer might depend on a and b.)

(b) Let $a = b = \frac{1}{2}$. Sketch a phase diagram covering the set where $0 \leq x \leq 2$, $0 \leq y \leq 2$, and indicate some solution curves.

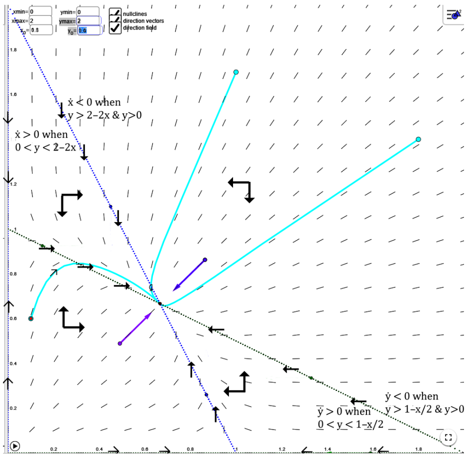
On problem 3

(a)
- Solve the equation system $(1 - ay - x) \cdot x = 0 = (1 - bx - y) \cdot y$: Either $x = 0$ or $x = 1 - ay$; AND, either $y = 0$ or $y = 1 - bx$. Four possibilities: (i) $(x, y) = (0, 0)$, (ii) $x = 0$ and $y = 1 - bx = 1$ yields $(x, y) = (0, 1)$, (iii) $y = 0$ and $x = 1 - ay = 1$ yields $(x, y) = (1, 0)$ and finally the point $(\bar{x}, \bar{y})$, which yields a linear equation system to solve. E.g., put $\bar{x} = 1 - a\bar{y} = 1 - a(1 - b\bar{x}) = 1 - a + ab\bar{x}$, so $\bar{x} = \frac{1-a}{1-ab}$, and calculate $\bar{y}$ to get $(\bar{x}, \bar{y}) = (\frac{1-a}{1-ab}, \frac{1-b}{1-ab})$.

  (It is not required to point out that $\bar{x}$ and $\bar{y}$ are positive. But for each, the assumptions grant that numerator and denominator have the same sign.)
- The Jacobian is $\begin{pmatrix} 1-ay-2x & -ax \\ -by & 1-bx-2y \end{pmatrix}$, which becomes $\begin{pmatrix} -\bar{x} & -a\bar{x} \\ -b\bar{y} & -\bar{y} \end{pmatrix}$ when $1 - a\bar{y} - \bar{x} = 1 - b\bar{x} - \bar{y} = 0$. The trace is negative, while the determinant is $\bar{x}\bar{y}(1 - ab) \neq 0$ as $ab \neq 1$ by assumption. (Inserting the coordinates yields the same, of course.)

  Therefore: if $ab > 1$, the determinant is negative and $(\bar{x}, \bar{y})$ is a saddle point, while if $ab < 1$ then (by trace<0), $(\bar{x}, \bar{y})$ is locally asymptotically stable.

(b) With $a = b = \frac{1}{2}$, the nullclines $x = 1 - ay$ and $y = 1 - ax$ become the lines $y = 2 - 2x$ (for $\dot{x} = 0$) and $y = 1 - \frac{1}{2}x$ (for $\dot{y}$). Each axis is a nullcline too, but in the set specified in the problem, they do not need to indicate anything there (though the diagram next page does). The ambition is to get things qualitatively right (correct straight-line nullclines, correct arrows at nullclines, solution curves consistent with that, ...). Those who get the wrong classification for the stationary state $(\bar{x}, \bar{y})$ for this case in part (a), may carry that error with them; the grading committee should exercise its best judgement. Example sketch:



Problem 4 Consider for constants $x_0 > 0$ and $T > 0$ the optimal control problem

$$\max_{u(t)\in[0,1]} \int_0^T (u - x^2)\,dt \qquad \text{where } \dot{x} = x + u, \quad x(0) = x_0 \quad \text{and } x(T) \text{ free.}$$

(a) State the conditions from the maximum principle. Are these conditions also sufficient?

(b) Show that an optimal control u^* must be 0 or 1 somewhere in the open interval $(0, T)$. (I.e., that it *cannot* be optimal to choose a u s.t. $u(t) \in (0, 1)$ for all $t \in (0, T)$.)

On problem 4

(a) Let $H(t, x, u, p) = u - x^2 + p \cdot (x + u)$. Since H is concave wrt. (x, u), the conditions to follow are sufficient as well:

i) u^* maximizes $u - x^2 + p \cdot (x + u)$ over $u \in [0, 1]$.

ii) p satisfies $\dot{p}(t) = -\frac{\partial}{\partial x}[u - x^2 + p \cdot (x + u)] = 2x - p$, with $p(T) = 0$.

iii) (The book does not include that the differential equation $\dot{x} = x + u$ (with $x(0) = x_0$) must hold, but they are free to include it.)

They can write out i) as $u^* = 1$ if $1 + p > 0$ and $u^* = 0$ if $1 + p < 0$. It is not necessary to state e.g. «there must exist a continuous, piecewise C^1 function p ...».

(b) Since $p(T) = 0$, then $p(t)$ must be > -1 for all sufficiently large $t < T$. For those t, we must have $u^*(t) = 1$. (Pointing out that $p(T) > -1 \Rightarrow u^*(T)$ `is one` (merely at T) does still apply the relevant condition from the maximum principle, which *is* the main point of the question, and should be awarded partial score accordingly.)

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2019:
The typewriter-font
"is one"
corrects a typo.
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Problem 5 Define the functions u and v on the (convex!) set $\{(x, y);\ x \geq 0,\ y \geq 0\}$ by

$$u(x, y) = (16xy)^3 \qquad \text{and} \qquad v(x, y) = x + \sqrt{x^2 + 2y}$$

(a) Decide quasiconcavity/quasiconvexity of each of the functions u and v.
Hints: (I) "neither" is wrong answer! (II) solve level curves $v(x, y) = C$ for y.

Consider now the *necessary* Kuhn-Tucker conditions – *disregarding* constraint qualifications, which you can take for granted that hold – associated to each of the problems

$$\max\ u(x, y) \quad \text{such that} \quad v(x, y) = 1, \quad x \geq 0, \quad y \geq 0 \tag{P1}$$

$$\max\ (-v(x, y)) \quad \text{such that} \quad u(x, y) \geq 1, \quad x \geq 0, \quad y \geq 0 \tag{P2}$$

It is a fact that $(x_1, y_1) = (\frac{1}{2}, 0)$ satisfies the necessary conditions associated to (P1), and that $(x_2, y_2) = (\frac{1}{4}, \frac{1}{4})$ satisfies the necessary conditions associated to (P2).

(b) For each problem (P1), resp. (P2), and the corresponding point (x_1, y_1), resp. (x_2, y_2):

Does the point (x_1, y_1) resp. (x_2, y_2) also satisfy *sufficient* Kuhn–Tucker conditions? If not: which part of the conditions fails?

Hint: The Lagrangians are *not* concave.

On problem 5 A general note: Hint (I) of (a) is to mitigate errors spilling over to part (b). With a hypothetical "neither" answer to part (a), one could argue that one is nowhere close to a sufficient condition in part (b). For part (b), some consistency with the answer to part (a) is expected; in particular, some would likely answer only "quasiconcave" or "quasiconvex" to the v function.

(a) u: one can calculate level curves $y = \text{constant}/x$ to find out it is quasiconcave; one can note it is an increasing transformation of a concave function (they are allowed to know that Cobb–Douglas with degree of homogeneity $\in (0, 1]$ is concave) – or one can also calculate the bordered Hessian determinant, which is practically feasible. Anyway, <u>u is quasiconcave.</u>

v on the other hand, is not an increasing transformation of a concave function, hence the hint (which was also employed in a seminar problem which only had a sign different). Besides, the bordered Hessian determinant involves some ugly calculations where it is easy to make mistakes. (And, the bordered Hessian determinant is zero – some might get confused and not spot that this does indeed imply the conclusion, which has not been stressed much in class.)
Proceeding to hint (II), $v(x, y) = C \Leftrightarrow \sqrt{x^2 + 2y} = C - x$, and squaring (for $x \leq C$) yields $2y = C^2 - 2x$, a straight line (needless to point out: intersected with the domain of definition). $v \leq C$ on/above the line (convex set!) and $\leq C$ on/below the line (convex). So <u>v is both quasiconcave and quasiconvex</u>.

(b) This question tests knowledge of quasiconcavity-based sufficient conditions, where the sufficient conditions yield no conclusion if the candidate point is stationary for the objective.

(P1) The point (x_1, y_1) is a stationary point for u, so sufficient conditions do not apply (and the point is not optimal, but that was not asked).
Those who got the correct answer that v is both quasiconcave and quasiconvex, should not claim that the sign of the multiplier matters, as λv is quasiconcave and quasiconvex for both signs. Those who inherited errors from part (a) should not be penalized in part (b) unless the error destroyes the problem to be solved.

(P2) The $-v$ and u are both quasiconcave (that means if rewritten with «$\leq$» constraint(s): quasiconcave objective and quasiconcave functions in the $\leq$ constraints), and furthermore, $\nabla v(x_2, y_2)$ is not null (indeed, v has no stationary points). So sufficient conditions apply and (x_2, y_2) solves problem (P2).

(End of grading guideline. Attached: two pages from the grading committee.)

Grading ECON4140 spring 2018

These are notes from the grading committee's work, two pages.

- We deviated from the default uniform weighting. Problem 5 was cut down to low weight (significantly benefiting the overall score), while Problem 1 was subject to a slight reduction in order not to overweight linear algebra too much. In the end, these adjustments did have some impact on score, but only on a very few grades (none for the worse).
- We also stretched the "A" threshold a bit downwards.

This note addresses some of the considerations made, and in particular some of the common errors – the amount of elementary theoretical shortcomings is surprising, taking into account that the course is no longer compulsory for any master programme. Scores stated are averaged over the passing grades, fails removed; grand average was slightly above 60 percent.

Problem 1

Problem 1 averaged to middle "B" score.

(a) Part (a) had a parameter-dependent rank, and a definiteness question. Some errors were expected – like the need to symmetrize for definiteness – but there was a major surprise: a *large majority* of the papers got rank right *only iff* $c \neq 0$. (Often, $\text{rank}(\mathbf{A}_0)$ was claimed to be 1 – wrong, and without any attempted justification.)
There was no sign that the difficulties slowed down the candidates (maybe they *should* have spent another minute or two?) and overall, part (a) got a middle "C" score.

(b) "A" for everyone!

(c) Median score = full score. Still, a few did not compute correctly ...

(d) ... and those who did *not* write out $p(\lambda)$ in (c), by and large were unable to calculate the determinant. They tried (hence knew the concept), but were unable to do the cofactor expansion properly.
Furthermore, more than one paper put c equal to some arbitrary number and got $\mu =$ that number. That is *not* independent of c.
Nevertheless, this part averaged nearly at the "B/C" threshold.

Problem 2

Problem 2 (a) and (b) averaged together to middle "B" score. Simple mistakes include going from $\cos\theta = 1/2$ to $\cos\frac{t}{2}$ – and in part (b) keeping one of the constants at zero, inherited from (a).

Part (c) – the high school exam problem – averaged less than the pass mark of 40 percent. Only a quarter of the candidates made a decent attempt at the induction step.

Problem 3

(a) Although every paper found at least one (correct or incorrect) point, about half of the passing candidates (and the fails!) missed one equilibrium point or more – usually by *dividing by zero.* The issue is not that these papers were blank about the concept of stationary states, but the median Mathematics 3 candidate is unable to solve the equations $(1-ay-x)\cdot x = 0 = (1-bx-y)\cdot y$ for (x,y) (despite trying hard enough to find at least one correct or incorrect solution point).
Some who failed to find $(\bar{x},\bar{y})$, but classified a different one. Some score awarded to the extent they showed ability to do what the second bullet point set out to test, and so part (a) ended up in the high sixties score, better than the problem set average.

(b) "F" score for the majority and on the average. Sketching phase diagrams could be time-consuming, and we can only conjecture that some gave priority to other questions, given the time budget.

Problem 4

(a) Averaged to a good "B" (closing in on "A"). Two common mistakes were seen this semester (too): confusing maximum with stationary point, and forgetting all about *bivariate* concavity – the infamous "$AC - B^2$" from ECON2200.

(b) Averaged to the pass mark, despite nearly half the papers scoring zero.

Problem 5

Worst in show, scoring 13 to 14 percent and no better among passing candidates than failing. Problem 5 had elements one could expect to come out as hard, but still there were negative surprises, in particular that the median Mathematics 3 student does the ECON2200 "$AC - B^2$" wrong (again!), and no-one could spot a transformed Cobb-Douglas.

(a)
- u: The *majority* of the papers *tried and failed* to calculate the sign of the Hessian determinant. Likely a "biased statistic": (i) wishful thinking, as some obviously *wanted* the determinant to be positive in order to claim concavity or convexity; (ii) candidates who submitted "blank" could have gotten a negative Hessian and chosen not to submit it (only one got the correct sign and pointed out it was not helpful).
  There was a bordered Hessian to be seen, but nobody identified u as a strictly increasing transformation of a Cobb–Douglas.
- v: All sorts of errors, including: (i) Squaring a sum to the sum of squares; (ii) Claiming that the square root is a convex function; (iii) Claiming that the square root of a convex function must be concave.
  And even candidates who were able to calculate the level curves algebraically, failed to conclude.

(b) Only a couple of papers knew about quasiconcave programs.