UNIVERSITY OF OSLO
DEPARTMENT OF ECONOMICS

Exam: **ECON4140 – Mathematics 3: Differential equations, static and dynamic optimization**

Date of exam: Friday, May 29, 2015 **Grades are given: June 10, 2015**

Time for exam: 2.30 p.m. – 5.30 p.m.

The problem set covers 2 pages of text (not incl. the cover sheet)

Resources allowed:

- All written and printed resources, as well as calculator, is allowed

The grades given: A-F, with A as the best and E as the weakest passing grade. F is fail.

University of Oslo / Department of Economics **(English only)**

ECON4140 Mathematics 3

May 29th 2015, 1430–1730.

There are 2 pages of problems to be solved.

All printed and written material may be used, as well as pocket calculators.

Grades given run from A (best) to E for passes, and F for fail.

- You are required to state reasons for all your answers.
- You are permitted to use any information stated in an earlier part or item (e.g. "(a)" or "i)") to solve a later one (e.g. "(c)" or "ii)"), regardless of whether you managed to answer the former. A later part or item does not necessarily utilize answers from or information given in a previous one.

Problem 1 Define for each $h \in \mathbf{R}$ the following matrices

$$\mathbf{A}_h = \begin{pmatrix} 5-h & 3 \\ 3 & 4-h \\ 2 & 3 \end{pmatrix}, \quad \mathbf{b}_h = \begin{pmatrix} 2 \\ 3 \\ 5-h \end{pmatrix}, \quad \mathbf{C}_h = \begin{pmatrix} 5-h & 3 & 2 \\ 3 & 4-h & 3 \\ 2 & 3 & 5-h \end{pmatrix}, \quad \mathbf{M} = \mathbf{C}_0$$

(where $\mathbf{C}_0$ denotes $\mathbf{C}_h$ with $h = 0$). Observe that $\mathbf{C}_h = \mathbf{M} - h\mathbf{I} = \big(\mathbf{A}_h | \mathbf{b}_h\big)$.

(a) $\mathbf{u} = (1, -2, 1)'$ is an eigenvector of $\mathbf{M}$. Find a corresponding eigenvalue λ_1. (You shall obtain that $0 < \lambda_1 < 3$.)

(b) $\lambda_2 = 3$ is an eigenvalue of $\mathbf{M}$. Find a corresponding eigenvector $\mathbf{v}$. (You shall obtain an answer such that $v_1 v_3 < 0$.)

(c) It is a fact that $\mathbf{M}$ has an eigenvector $\mathbf{w}$ with all coordinates nonnegative. Show why this fact together with parts (a) and (b) imply that $\mathbf{M}$ must be positive definite. (You are required to use precisely these pieces of information; you will not be rewarded for using other calculations.)

(d) Show that $\mathbf{A}_h$ has rank 2 no matter what h is.

(e) Decide whether the following statement is true or false: *"The equation system* $\mathbf{A}_h \binom{p}{q} = \mathbf{b}_h$ *has a solution* $\binom{p}{q}$ *if and only if h is an eigenvalue for* $\mathbf{M}$*."*

Problem 2 Given constants $r \geq 0$, $s > 0$ and $t > 0$, a vector $\mathbf{m} \in \mathbf{R}^n$ such that $1 = m_1 \geq m_2 \geq \ldots m_n \geq 0$, and for $\mathbf{x} \in \mathbf{R}^n$ the functions

$$g(\mathbf{x}) = |x_1| + \ldots + |x_n|, \qquad F(\mathbf{x}) = \mathbf{m}'\mathbf{x} - sg(\mathbf{x}) + (s-1)t, \qquad H(\mathbf{x}) = F(\mathbf{x}) - r \max_i |x_i|$$

(where $\max_i |x_i|$ means the greatest of the n numbers $|x_1|, \ldots, |x_n|$).

(a) i) Show that H is concave for every $r \geq 0$, $s > 0$.
ii) Consider part (b) below. Explain why the existence of such an s as asked for in part (b), will show that $\mathbf{x}^* = (t, 0, \ldots, 0)'$ solves the nonlinear programming problem

$$\max_{\mathbf{x}} \ \mathbf{m}'\mathbf{x} \quad \text{subject to} \quad g(\mathbf{x}) \leq t$$

(b) Find an $s \in [0, 1]$ such that $\mathbf{0}$ is a supergradient for F at $\mathbf{x}^* = (t, 0, \ldots, 0)'$.
Hint: Explain why it suffices to show that F attains a (local or global) maximum at $\mathbf{x}^*$, and then show that this happens for some $s \geq 0$. You shall get that $m_n \leq s \leq m_1$ and also that s does not depend on t (if you need to, check the case $t = 1$ first).

Problem 3 Let $0 < K < Q < 1$ be constants and let G be a given function. Consider the differential equation system

$$\begin{aligned} \dot{x}(t) &= p(t) + Q \\ \dot{p}(t) &= Kx(t) - G(t) \end{aligned} \tag{D}$$

(a) Deduce a second-order differential equation for x, and find the general solution of this equation when $G \equiv 0$. (*Hint:* For which γ will $x(t) = e^{\gamma t}$ be a particular solution?)

(b) Find the general solution of (D) for the case when $G(t) = Ke^t$.

Problem 4 Let $0 < K < Q < 1$ be constants, and consider the optimal control problem

$$\max_{u(t) \in \mathbf{R}} \int_0^{11} \left\{ -\frac{K}{2} \cdot \left[x(t) - e^t\right]^2 - \frac{1}{2}\left[u(t)\right]^2 \right\} dt, \qquad \dot{x} = u + Q, \quad x(0) = x_0, \quad x(11) \text{ free.}$$

(a) i) State the conditions from the maximum principle.
ii) Are these conditions also sufficient?

(b) Show that in optimum, x and the adjoint (costate) p must satisfy the differential equation system (D) in problem 3, with $G(t) = Ke^t$.

(c) Suppose that for some set of parameters the optimal solution ends at $x(11) = 11e^{11}$. Approximately how much would the optimal *value* change if the final time were reduced from 11 to 10.9?