

UNIVERSITY OF OSLO
DEPARTMENT OF ECONOMICS

Exam: ECON4140 – Mathematics 3: Differential Equations, Static and Dynamic Optimization

Date of exam: Friday, June 10, 2016

Grades are given: June 24, 2016

Time for exam: 09.00 a.m. – 12.00 noon

The problem set covers 3 pages (incl. cover sheet)

Resources allowed:

- All written and printed resources – as well as calculator - is allowed

The grades given: A-F, with A as the best and E as the weakest passing grade. F is fail.

ECON4140 Mathematics 3

June 10th 2016, 0900–1200.

There are 2 pages of problems to be solved.

All printed and written material may be used, as well as pocket calculators.

Grades given run from A (best) to E for passes, and F for fail.

- You are required to state reasons for all your answers.
- You are permitted to use any information stated in an earlier letter-enumerated item (e.g. “(a)”) to solve a later one (e.g. “(c)”), regardless of whether you managed to answer the former. A later item does not necessarily require answers from or information given in a previous one.

Suggested weighting: Problems 1, 2, 3 and 4 are *expected* to count 25 percent each.**Problem 1** For arbitrary real constants a, b, c, d, p, q, r, s , define the matrices

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} a & 0 & 0 & b \\ 0 & p & q & 0 \\ 0 & r & s & 0 \\ c & 0 & 0 & d \end{pmatrix}, \quad \mathbf{G} = \begin{pmatrix} a & 0 & 0 & b & a & b \\ 0 & p & q & 0 & c & d \\ 0 & r & s & 0 & a & b \\ c & 0 & 0 & d & c & d \end{pmatrix}$$

Notice that \mathbf{F} has \mathbf{B} as the “middle” block and the elements of \mathbf{A} in the corners, and that the blocks of \mathbf{G} are $(\mathbf{F}; \mathbf{A})$. Let prime (e.g. \mathbf{F}') denote transpose.

- (a)
 - Show that if $qr = 0$, then either $(0, 1, 0, 0)'$ or $(0, 0, 1, 0)'$ is an eigenvector of \mathbf{F} , and
 - find the eigenvalue corresponding to either $(0, 1, 0, 0)'$ or $(0, 0, 1, 0)'$.
- (b) If $qr = ps \neq 0$, then $p + s$ is an eigenvalue of \mathbf{F} .
- Find a corresponding eigenvector of the form $(h, k, \ell, 0)'$.
- (c) Show that for λ to be an eigenvalue of \mathbf{F} , then λ must be an eigenvalue of \mathbf{A} or an eigenvalue of \mathbf{B} . (Or possibly both. *Hint:* what is the product of the characteristic polynomials of \mathbf{A} and \mathbf{B} ?)

Problem 2

(a) Solve the difference equation

$$x_{t+2} + \frac{1}{6}x_{t+1} - \frac{1}{6}x_t = 2 \quad (\text{D})$$

(b) • Show that the differential equation system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 6 \end{pmatrix} \quad (\text{S})$$

implies that x satisfies

$$\ddot{x} + \frac{1}{6}\dot{x} - \frac{1}{6}x = 2 \quad (\text{E})$$

- Find the general solution of (E)
- (c) • Is (D) stable?
- Is (E) stable?
- Classify the equilibrium point $(x^*, y^*) = (-12, 6)$ of (S).

Problem 3 Let $a_t > 0$, $b_t > 0$, $m > 1$, $T > t_0$ and $x_{t_0} > 0$ be given numbers. Consider the dynamic programming problem

$$J_{t_0}(x_{t_0}) = \max_{u_t \in [0,1]} \sum_{t=t_0}^T a_t^{1/m} \cdot ([u_t + b_t] \cdot x_t)^{1-1/m}, \quad \text{subject to } x_{t+1} = x_t \cdot (1 - u_t)$$

Although you cannot use without proof the information given in part (b) in part (a), it gives a hint that is likely helpful.

- (a) In this part, let $a_t = b_t = 1$ for all t . Calculate J_T , J_{T-1} and J_{T-2} .
- (b) Show by induction that the optimal u_t^* does not depend on x .

Problem 4 Let r , k , T and x_0 be given numbers, all > 0 , and let n be a natural number ($\in \{1, 2, 3, \dots\}$). Consider the optimal control problem

$$\max_{u(t) \geq 0} \int_0^T [kx(t) + e^{-rt}(u(t))^{1-1/2n}] dt, \quad \text{subject to } \dot{x} = -u, \quad x(0) = x_0, \quad x(T) \geq 0.$$

- (a) • State the conditions from the maximum principle. (Put the « p_0 » constant = 1.)
- Do we know that a pair (x^*, u^*) satisfying these conditions, will be optimal?
- (b) Show that there are constants α , β and γ such that
 - the optimal control is of the form $u^*(t) = (\alpha + \beta \cdot (t - T))^{-2n} e^{-\gamma t}$, with
 - $\alpha \geq 0$.