UNIVERSITY OF OSLO
DEPARTMENT OF ECONOMICS

Exam: **ECON4140 – Mathematics 3: Differential Equations, Static and Dynamic Optimization**

Date of exam: Wednesday, May 24, 2017 **Grades are given:** June 14, 2017

Time for exam: 2.30 p.m. – 5.30 p.m.

The problem set covers 4 pages (incl. cover sheet)

Resources allowed:
- Open book exam. All written and printed resources – as well as calculator - is allowed

The grades given: A-F, with A as the best and E as the weakest passing grade. F is fail.

University of Oslo / Department of Economics **English version only**

ECON4140 Mathematics 3

May 24th 2017, 1430–1730.

There are 3 pages of problems to be solved (including a figure on the last page).

All printed and written material may be used, as well as pocket calculators.

Grades given run from A (best) to E for passes, and F for fail.

- You are required to state reasons for all your answers.
- You are permitted to use any information stated in an earlier letter-enumerated item (e.g. "(a)") to solve a later one (e.g. "(c)"), regardless of whether you managed to answer the former. A later item does not necessarily require answers from or information given in a previous one.
- "Suggested" weights: the grading committee is free to deviate.

Problem 1 *Suggested weight: 25 %.* In this problem, let $\mathbf{A} = \begin{pmatrix} 4 & 1 \\ 2 & -3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -4 & 1 \\ 2 & -3 \end{pmatrix}$ and let (S) be the nonlinear differential equation system

$$\begin{aligned} \dot{x} &= y + x^4 - 1 \\ \dot{y} &= x^3 - x - 3y \end{aligned} \tag{S}$$

(a) Consider in this part each of the differential equation systems $\dot{\mathbf{w}} = \mathbf{A}\mathbf{w}$ and $\dot{\mathbf{z}} = \mathbf{B}\mathbf{z}$: Decide whether it is globally asymptotically stable or unstable – and if unstable, whether the origin is a saddle point or not.

(b) The system (S) has precisely two equilibrium points, both on the x axis. Find and classify these. (*Hint:* The linear systems from part (a) will show up.)

(c) Sketch a phase plane for the system (S) and indicate some particular solution curves. (*Hint:* The diagram on the last page gives a hint.)

Problem 2 *Suggested weight: 15 %.* Consider the problem

$$J_t(x_t) = \max \left\{ \ln x_T + \sum_{s=t}^{T-1} \left(\ln x_s + (1 - u_s)\sqrt{x_s} \right) \right\} \qquad \text{where } x_{t+1} = x_t u_t, \qquad u_t > 0.$$

- Use dynamic programming to calculate $J_{T-1}(x)$ and $J_{T-2}(x)$. .

You can express $J_{T-2}(x)$ in terms of the function $w(x)$ satisfying $w = 1/\sqrt{u^*_{T-2}}$, as long as you obtain a formula for w.

Problem 3 *Suggested weight: 20 %.* Let x_0, T and k be constants, all > 0, and let m be a real constant. Consider the following optimal control problem:

$$\max_{u(t)\geq m} \int_0^T \left([x(t)]^2 - u(t)\cdot[x(t)]^{k+1}\right) dt \qquad \text{where } \dot{x}(t) = u(t)x(t) \text{ and } x(0) = x_0.$$

Terminal state $x(T)$ is free, but note that $x(T) > 0$ because $x(t) \geq x_0 e^{mt}$ for all $t > 0$.

(a)
- State the conditions from the maximum principle, and
- show that these conditions imply that $u(T) = m$, i.e. the left endpoint of U.

(b) Let now $m = 0$, so the control region is $[0, \infty)$.
Show that if $T < \frac{1}{2}x_0^{k-1}$, the control $u \equiv 0$ – i.e., so that $(x(t), u(t)) = (x_0, 0)$ for all t – satisfies all the conditions from the maximum principle.

Problem 4 *Suggested weight: 40 %.* Define the matrices $\mathbf{M}$ and $\mathbf{H}$ and the vectors $\mathbf{h}_i$ as follows; observe that the $\mathbf{h}_i$ are the columns of $\mathbf{H}$:

$$\mathbf{M} = \begin{pmatrix} 2 & -12 & 3 \\ 2 & -9 & 2 \\ 6 & -24 & 5 \end{pmatrix}, \quad \mathbf{H} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 4 \\ -1 & 4 & 17 \end{pmatrix}, \quad \mathbf{h}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \mathbf{h}_2 = \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix}, \quad \mathbf{h}_3 = \begin{pmatrix} -1 \\ 4 \\ 17 \end{pmatrix}$$

(a)
- Show that $\{\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3\}$ form a linearly *dependent set*, and find the rank of the matrix $\mathbf{H}$.
- Show that each $\mathbf{h}_i$ is an eigenvector of $\mathbf{M}$, with the same eigenvalue λ.

(b) In the first bullet item, you shall find an eigenvalue $\mu \neq \lambda$, but for full score you shall be able to do so without calculating the characteristic polynomial:
- Find μ using the following fact (which you need not show): *the rank of* $\mathbf{M}$ *is two.* Half score on this bullet item may be obtained by instead finding μ by calculating (or starting to calculate) the characteristic polynomial.
- Find an eigenvector $\mathbf{v}$ associated with μ.

(c) The functions $Q(\mathbf{x}) = \mathbf{x}'\mathbf{H}\mathbf{x}$ and $R(\mathbf{x}) = \mathbf{x}'\mathbf{M}\mathbf{x}$ are quadratic forms. For each of these, decide whether it is indefinite or positive/negative definite/semidefinite.

(d) Let $\mathbf{x}_0$ be a given vector, and define for each nonnegative integer t

$$\mathbf{x}_{t+1} = \mathbf{M}\mathbf{x}_t + \pi\mathbf{h}_1 + \mathbf{v} \qquad (\text{where } \pi = 3.14159...)$$

You can take for granted that each $\mathbf{x}_t$ can be written as $\mathbf{x}_t = a_t\mathbf{h}_1 + b_t\mathbf{h}_2 + c_t\mathbf{v}$ where $\mathbf{h}_1$, $\mathbf{h}_2$ and $\mathbf{v}$ are the above eigenvectors of $\mathbf{M}$.

- *Use induction* to establish a linear first-order difference equation $a_{t+1} = ka_t + d$ for the a_t coefficients, and find the constants k and d.
(*Hint:* The "base case" says that a_0 exists, which you can take for granted is true.)

Appendix: hint for problem 1 part (c):