University of Oslo / Department of Economics / NCF

ECON4140 Mathematics 3 – on the 2019–05–20 exam

- *Standard disclaimer:*
  - This note is not suited as a complete solution or as a template for an exam paper. It was written as guidance for the grading process – however, with additional notes and remarks for using the document in teaching later.
  - The document reflects what was expected in that particular semester, and which may not be applicable to future semesters.
- *Weighting:* The problem set was written with the intention that a uniform weighting over letter-enumerated items should be a *feasible* choice. The grading committee – and in case of appeals, the appeals committee – can decide otherwise.
  - Problem 1 covers two topics, namely linear algebra and differential equations. The apparently high weight (5/11 if uniform) for Problem 1, was intentional.
- *Default grade intervals:* Default percent score to grade conversion table for this course:

F (fail)	E	D	C	B	A
0 to 39	40 to 44	45 to 54	55 to 74	75 to 90	91 to 100

  The committee (and in case of appeals, the new committee) is free to deviate.
- *Recent "deviations from defaults" for reference:* 2016 through 2018 have had post-grading information available, indicating deviations from weighting and conversion table and other measures made. Brief overview with links to documents provided:
  - 2016: Weighting and conversion table kept at default, but certain elementary errors and misconceptions got a more forgiving treatment.
  - 2017: The committee considered the exam a bit easy, although the threshold for A *«was practiced slightly leniently in order to distinguish out the best»*.
  - 2018: Weighting was tweaked (for the benefit of a very few papers) and the committee *«stretched the "A" threshold a bit downwards»*.

Addendum after grading: The issues remarked on 1(c) did materialize. Both a «generous treatment» and zero-weighting were considered. «Default» grade thresholds were applied.

Notes The next pages will restate problems 1–4 as given, each followed by annotated solution. A particular note on references to the 2018 exam: as has become a tradition, the final review – this time May 16th, four days before the exam – covered the previous (ordinary) exam. A few common issues from the grading of that exam (highlighted in the 2018 guideline), were addressed specifically; and, in particular in relation to the above «Standard disclaimer», pointing out that this year was laid out a bit different than 2018. It was intention to reduce the risk that some will take the 2018 document as general policy, but some might still have considered quasiconcavity/quasiconvexity (problem 4) to be on the edge problems. Notes are in particular given for 1(c) and 4.

Problem 1 Note, this problem involves more than one topic. Parts (c) and (d) require you to use *only* the information given in (a) and (b), but you can solve part (e) by use of any means you wish.

Let $\mathbf{M} = \begin{pmatrix} 7 & 2 \\ 16 & 3 \end{pmatrix}$.

(a) $\mathbf{w} = (1 \;\; 2)'$ is an eigenvector of $\mathbf{M}$. Find the corresponding eigenvalue $\mu > 0$.

(b) Find an eigenvalue $\lambda < 0$ and a corresponding eigenvector $\mathbf{v}$.

(c) What can parts (a) and (b) tell us about the definiteness of the quadratic form $q(x, y) = (x,\ y)\mathbf{M}\binom{x}{y}$? If applicable: what information/property would be missing? (*Hint/warning:* mind the details. You are required to use only parts (a) and (b).)

(d) What can parts (a) and (b) tell us about the stability property of the differential equation system $\dot{\mathbf{z}} = \mathbf{M}\mathbf{z}$? If unstable, can (a) and (b) tell whether any non-constant particular solution converges? (Again, you are required to use only parts (a) and (b).)

(e) Let $\mathbf{w} = \binom{1}{2}$ as in part (a) and h be a given continuously differentiable function. Consider the differential equation system $\binom{\dot{x}}{\dot{y}} = \mathbf{M}\binom{x}{y} + h(t)\mathbf{w}$.

- Deduce a second-order differential equation for x (for general h), and
- find a particular solution of that equation if $h(t) = e^{\pi t}$, and
- explain how to find a particular solution if instead $h(t) = t^{2019}$.

How to solve, and grading notes: This note starts at (c) and (d) to highlight how they only require information *given in the (a) and (b) problem text* – they do not require solving (a) and (b). Also, there are notes to be made on (c) in particular.

(c) The 2018 version did not stress anything about what could be inferred when the matrix is not symmetric, and the grading guidelines for 2018 mentioned that it was not required. This year's class has been informed that the non-symmetric matrix can *rule out* definiteness properties; nevertheless, the committee must be aware the risk that the problem question could invite the answer «nothing, $\mathbf{M}$ is not symmetric». Such an answer would in any case be worth a partial score (they are expected to know that symmetry matters!), and the committee might consider to adjust partial scores upwards due to earlier versions – including 2018 – not stressing this at all.

It was certainly intentional to write «What can parts ...» to suggest that there could be partial information, as there will always be when an eigenvalue sign is given. Indeed, here we can rule out pos./neg. semidefiniteness and conclude completely.

To solve part (c): From (a), $q(\mathbf{w}) = \mathbf{w}'\mathbf{M}\mathbf{w} = \mu||\mathbf{w}||^2 > 0$, so q attains positive values (ruling out negative semidefiniteness). From (b) we know that $q(\mathbf{v}) = \mathbf{v}'\mathbf{M}\mathbf{v} = \lambda||\mathbf{v}||^2$ for some (eigenvector, hence non-null) $\mathbf{v}$. So q attains negative values as well, since $\lambda < 0$. As q attains both signs, we can conclude that $\underline{\underline{q \text{ is indefinite}}}$.

A solution based on symmetrization but still using only parts (a) and (b), could go as follows: q has the same definiteness property as $\mathbf{M}' + \mathbf{M}$. From (a), $\mathbf{w}'(\mathbf{M}' + \mathbf{M})\mathbf{w} = (\mathbf{w}'\mathbf{M}')\mathbf{w} + \mathbf{w}'\mu\mathbf{w} = (\mathbf{M}\mathbf{w})'\mathbf{w} + \mu||\mathbf{w}||^2 = 2\mu||\mathbf{w}||^2 > 0$ and similar for $\mathbf{v}$.

(d) Opposite-sign real eigenvalues $\Rightarrow$ saddle point: unstable, and with two convergent non-constant solution paths. (They were not asked to *find* those.)

Note that (c) and (d) only require the problem text of (a) and (b). Their solutions:

(a) Calculate $\mathbf{M}\mathbf{w} = \binom{11}{22}$ and identify it as a scaling $\underline{\underline{11}}$ of $\mathbf{w}$.

(b) Eigenvalues sum to trace, so $\lambda = 7+3-\mu = 7+3-11 = \underline{\underline{-1}}$. To find an eigenvector: $\mathbf{M} - \lambda\mathbf{I} = \left(\begin{smallmatrix} 8 & 2 \\ 16 & 4 \end{smallmatrix}\right)$, and $\mathbf{v}$ satisfies $(8,2)\mathbf{v} = 0$. Any nonzero scaling of $\underline{\underline{(-1,4)'}}$ will do.

Note, if they solve out the eigenvalue from the characteristic polynomial: they are allowed to know without deducing, that when $n = 2$ then $p(\lambda) = \lambda^2 - \lambda\,\mathtt{tr}\,\mathbf{M} + \mathtt{det}\,\mathbf{M}$.

Finally, the differential equation for x:

(e) It is «known» – given in the book and and easy to memorize – that one is going to obtain $\ddot{x} - \dot{x}\,\mathtt{tr}\,\mathbf{M} + x\,\mathtt{det}\,\mathbf{M} = f(t)$ for some f, and it will likely be acceptable for the first bullet item to write the left-hand side of this out and only spend calculations on f:

- $\ddot{x} = 7\dot{x} + 2\dot{y} + \dot{h} = 7\dot{x} + 2(16x + 3y + 2h) + \dot{h} = 7\dot{x} + 32x + 4h + \dot{h} + 3\cdot(\dot{x} - 7x - h)$ so that $\underline{\underline{\ddot{x} - 10\dot{x} - 11x = h + \dot{h}}}$
- If $h(t) = e^{\pi t}$, the RHS is $(\pi+1)e^{\pi t}$. Try $Ke^{\pi t}$ and fit K: $K\cdot[\pi^2 - 10\pi - 11]e^{\pi t} = (\pi+1)e^{\pi t}$ yields a particular solution $\underline{\underline{\dfrac{\pi+1}{\pi^2 - 10\pi - 11}e^{\pi t}}}$ (They are not required to cancel down to $e^{\pi t}/(\pi - 11)$.)
- If $h(t) = t^{2019}$, the right-hand side is a 2019th degree polynomial. For a particular solution, try a general 2019th degree polynomial and fit coefficients. (The «2019» was chosen to discourage anyone from carrying it out in full detail.)

Problem 2 Consider the dynamic programming problem

$$J_{t_0}(x) = \max_{u_t > 0}\left\{x_T + \ln x_T + \sum_{t=t_0}^{T-1}(u_t + \ln u_t)\right\}, \qquad x_{t+1} = x_t - u_t \qquad \text{starting at } x_{t_0} = x > 0.$$

It is possible to start at part (b) and deduce (a) afterwards.

(a) Calculate J_{T-1} and J_{T-2}.

(b) Use induction to show that for each $s = 0, 1, \ldots$, we have $J_{T-s}(x) = x + C_{T-s}\cdot\ln\dfrac{x}{C_{T-s}}$ with $C_{T-s} > 0$ not depending on x.

(c) Consider the problem obtained in the limit $T \to +\infty$:

- State the associated Bellman equation.
- Why can we *not* expect the Bellman equation to have a (finite) solution $J(x)$? (*Hint:* Look at the limit of C_{T-s}.)

How to solve Problem 2: This note will do (b) then (c) then (a).

(b) The statement is true at time T i.e. $\tau = 0$, with $C_0 = 1$. For induction, assume true for τ. Then for $\tau + 1$ we have the following, denoting $c = C_{T-\tau}$:

$$J_{T-(\tau+1)}(x) = \max_{u>0} \Big\{ u + \ln u + J_\tau(x-u) \Big\} = \max_{u>0} \Big\{ u + \ln u + c \ln \frac{x-u}{c} + x - u \Big\}$$
$$= x + \max_u \Big\{ \ln u + c \ln \frac{x-u}{c} \Big\}$$

By the induction hypothesis, $c > 0$. A maximizing $u^* \in (0, x)$ will exist, and is given by the FOC $\frac{1}{u^*} = \frac{c}{x-u^*}$ so that $u^* = \frac{x}{1+c}$ and thus $\frac{x-u^*}{c} = \frac{x}{1+c}$ as well. Inserting:

$$J_{\tau-1}(x) = x + \ln \frac{x}{1+c} + c \ln \frac{x}{1+c}, \qquad \text{OK with } C_{T-\tau-1} = 1 + c = 1 + C_{T-\tau}.$$

(c) Because $C_{T-\tau} \to +\infty$ with T, the infinite horizon problem has a value of «$x + \infty \ln \frac{x}{\infty} = x + \infty \cdot \ln 0 = -\infty$», and so we cannot expect any solution to the Bellman equation $\underline{\underline{J(x) = \max_{u>0} \big\{ u + \ln u + J(x-u) \big\}}}$.

(a) From (b), $C_s = 1 + s$, so $\underline{\underline{J_{T-1}(x) = x + 2 \ln \frac{x}{2}}}$ and $\underline{\underline{J_{T-2}(x) = x + 3 \ln \frac{x}{3}}}$.

Notes for grading: A fairly similar seminar problem was assigned after Easter. There, a discount factor had to be corrected half a week before the seminar, and just in case it still messed up for someone, the above Problem 2 has no discounting.

No discounting is «why» the infinite-horizon problem has $-\infty$ value – and there is a possibility that the wording might be taken to invite an answer like «this problem has no discounting». Obviously there is insight in such an answer, but without context would be insufficient (consider a problem where optimum yields 0 running utility from the second period on). Should such answers show up, the committee should exercise best judgement.

The «can *not* expect» wording reflects that Mathematics 3 does not cover any precise conditions for when the Bellman equation is necessary/sufficient, so the question is not about disproving any «false solution candidate that satisfies the Bellman equation». It would be way good enough to point out that the DP problem has no solution (therefore none that can fit the Bellman criterion) – or to write that $C = C + 1$ would be impossible.

Problem 3 Let T, r and x_0 be constants, all > 0. Consider the variational problem

$$\min \int_0^T e^{-rt}\Big(x(t)^3 \cdot \dot{x}(t)\Big)^2\,dt, \qquad x(0) = x_0, \quad x(T) = 2.$$

(a)
- State the associated Euler equation.
- State the conditions from the maximum principle, obtained by rewriting as an optimal control problem (maximization!) with control $u = \dot{x} \in (-\infty, +\infty)$.

Let from now on $x_0 = T = 1$ and $r = \ln 2$. Take for granted that this $x^*(t)$ is optimal:

$$x^*(t) = \sqrt{3 \cdot 2^t - 2} \qquad \text{so that} \quad \dot{x}^*(t) = \frac{3\ln 2}{2} \cdot \frac{2^t}{x^*(t)} \tag{$*$}$$

Hint: It is possible to answer the following part (b) *without* solving any differential equation, if you use formulae $(*)$. You are *not* asked to show or verify $(*)$.

(b)
- Calculate $p(1)$, where p is the adjoint variable from the maximum principle. (You are allowed to calculate the current-value adjoint $\lambda(1)$ instead.)
- Find an expression for how much, approximately, the *optimal value* changes if T increases from 1 to $1 + 1/144$. (If you did not manage to solve the previous bullet item, use the number e in place of $p(1)$ or of $\lambda(1)$.)

How to solve; notes and guidelines:

(a)
- The derivative wrt. state is $e^{-rt} \cdot 6x^5(\dot{x})^2$, and the wrt. control: $e^{-rt}x^6 \cdot 2\dot{x}$. Total derivative of the latter: $2 \cdot \big[-re^{-rt}x^6\dot{x} + 6e^{-rt}x^5\dot{x}^2 + e^{-rt}x^6\ddot{x}\big]$. Equating: $\underline{\underline{3e^{-rt}x^5(\dot{x})^2 = -re^{-rt}x^6\dot{x} + 6e^{-rt}x^5\dot{x}^2 + e^{-rt}x^6\ddot{x}}}$ which, if so you prefer, can be simplified to $0 = -rx\dot{x} + 3\dot{x}^2 + x\ddot{x}$.
- To maximize $\int_0^T (-e^{-rt}) \cdot x^6u^2\,dt$ with $\dot{x} = u$ and $x(0)$ and $x(T)$ given, we get Hamiltonian $H(t, x, u, p) = -e^{-rt}x^6u^2 + pu$, and conditions:

  $$\underline{\underline{u^* \text{ maximizes } pu - e^{-rt}x^6u^2 \text{ over } u \in \mathbb{R}}} \quad \text{and}$$

  $$\underline{\underline{\dot{p} = 6e^{-rt}(x^*)^5(u^*)^2}} \quad \text{(no transversality condition)}$$

  or the equivalent current-value formulation if so they prefer: then u^* would maximize $\lambda u - x^6u^2$ where $\dot{\lambda} = r\lambda + 6x^5u^2$. One can hardly require it to stated that there is no condition on $p(T)$ (it is asked what conditions there are, not which ones there aren't), which is why it isn't double-underlined above.
  Other notes: They are free to include «$\dot{x}^* = u^*$». The degenerate case with $p_0 = 0$ need not be mentioned. Some would probably *solve out* for u^* by the FOC, which can hardly be penalized when the control region is open. And, it will be needed in part (b); for errors in maximization, the committee should exercise its best judgement considering (a) and (b) together.

(b) Notes first: The main points of part (b) are to apply the maximum principle to calculate u^* and p in terms of each other (this time solving out for $p(1)$) and to know the sensitivity result that the derivative wrt. T is the Hamiltonian evaluated at optimum at T. The theory is more important than getting the powers of two correct.

- From the maximum principle, $u^* = p/(2e^{-rt}x^6)$ so $p(1) = 2e^{-r}(x^*(1))^6 u^*(1)$. Inserting $2e^{-r} = 1$, $x^*(1) = 2$ and $u^*(1) = \frac{3}{2}\ln 2$ from $(*)$, yields $\underline{\underline{p(1) = 96\ln 2}}$. The current-value: $\lambda(1)$ would be twice this.
- Evaluate (at $T = 1$) the optimized Hamiltonian as $p(1)\cdot\frac{3}{2}\ln 2 - \frac{1}{2}2^6(\frac{3}{2}\ln 2)^2 = 3\cdot 2^5\cdot\frac{3}{2}(\ln 2)^2 - 2^5(\frac{3}{2}\ln 2)^2 = 72\cdot(\ln 2)^2$ and scale by 1/144: answer $= \underline{\underline{\frac{1}{2}(\ln 2)^2}}$. (Note, this is a «present-value question» and needs the discounting.)

Problem 4 Let ϕ be the strictly increasing function $\phi(z) = z^{1/3} + e^{z/3}$.
Decide the quasiconcavity/quasiconvexity(/both/neither) of the three functions

$$f(z) = \phi(z+1) + \phi(z-1), \qquad g(x,y) = \phi(y - x\cdot(1-x)) \qquad \text{and} \qquad h(x,y) = \phi(x^{3/2}y^e),$$

f and g defined everywhere and h for $x \geq 0$, $y \geq 0$ (all domains convex).

How to solve, and notes: The committee should exercise its judgement on whether it goes without saying – especially when this is the end of the exam – that we cannot have quasiconcavity for g and cannot have quasiconvexity for h.

f: f, like ϕ, is a strictly increasing function of a single variable (being the sum of two such), thus both quasiconcave and quasiconvex.

(«Sum of quasiconvexes» etc., is a fallacy: such one need not be quasiconvex. The counterexample given on the board was $\sqrt{|z+1|} + \sqrt{|z-1|}$.)

g: A strictly increasing transformation of the convex function $y - x + x^2$, is quasiconvex.

h: $x^{3/2}y^e$ is an increasing transformation of a Cobb–Douglas, hence quasiconcave. Being an increasing transformation of this, h is quasiconcave. Notes:

- The above justification would hold, given how well-known the Cobb–Douglas should be. See however next item. A more zealous argument would be that $\phi(x^{3/2}y^e)$ is an increasing transformation of $\ln(x^{3/2}y^e) = \frac{3}{2}\ln x + e\ln y$, which is a sum of concaves.
- The *2018* exam asked for the quasiconcavity of $(16xy)^3$, and nobody recognized it as a transformation of a concave Cobb–Douglas. It has been stressed this semester. Hopefully, the exam papers will offer justification of the quasiconcavity of $x^{3/2}y^e$, but chances are that there will both be students which skip justification because they cannot given any, and those who skip because they don't see the need. Exercise judgement; the sum $\frac{3}{2} + e$ of exponents is deliberately > 1, in order to trap those who claim concavity without checking.