UNIVERSITY OF OSLO
DEPARTMENT OF ECONOMICS

Exam: **ECON4140 – Mathematics 3: Differential Equations, Static and Dynamic Optimization**

Date of exam: Monday, May 20, 2019 **Grades are given:** June 7, 2019

Time for exam: 09.00 a.m. – 12.00 noon

The problem set covers 3 pages (incl. cover sheet)

Resources allowed:

- Open book exam. All written and printed resources – in addition to one out of two different calculators is allowed.

The grades given: A-F, with A as the best and E as the weakest passing grade. F is fail.

University of Oslo / Department of Economics **English version only**

ECON4140 Mathematics 3

May 20th 2019, 0900–1200.

There are 2 pages of problems to be solved.

All printed and written material may be used, as well as both the approved calculators.

Grades given run from A (best) to E for passes, and F for fail.

- You are required to state reasons for all your answers.
- You are permitted to use any information stated in an earlier letter-enumerated item (e.g. "(a)") to solve a later one (e.g. "(c)"), regardless of whether you managed to answer the former. A later item does not necessarily require answers from or information given in a previous one.
- Weighting: At the grading committee's discretion. The problem set is intended to facilitate equal weight over letter-enumerated items (Problem 4 counting as one).

Problem 1 Note, this problem involves more than one topic. Parts (c) and (d) require you to use *only* the information given in (a) and (b), but you can solve part (e) by use of any means you wish.

Throughout the problem, let $\mathbf{M} = \begin{pmatrix} 7 & 2 \\ 16 & 3 \end{pmatrix}$.

(a) $\mathbf{w} = (1\ \ 2)'$ is an eigenvector of $\mathbf{M}$. Find the corresponding eigenvalue $\mu > 0$.

(b) Find an eigenvalue $\lambda < 0$ and a corresponding eigenvector $\mathbf{v}$.

(c) What can parts (a) and (b) tell us about the definiteness of the quadratic form $q(x, y) = (x,\ y)\mathbf{M}\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$? If applicable: what information/property would be missing? (*Hint/warning:* mind the details. You are required to use only parts (a) and (b).)

(d) What can parts (a) and (b) tell us about the stability property of the differential equation system $\dot{\mathbf{z}} = \mathbf{M}\mathbf{z}$? If unstable, can (a) and (b) tell whether any non-constant particular solution converges? (Again, you are required to use only parts (a) and (b).)

(e) Let $\mathbf{w} = \left(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right)$ as in part (a) and h be a given continuously differentiable function. Consider the differential equation system $\left(\begin{smallmatrix} \dot{x} \\ \dot{y} \end{smallmatrix}\right) = \mathbf{M}\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) + h(t)\mathbf{w}$.

- Deduce a second-order differential equation for x (for general h), and
- find a particular solution of that equation if $h(t) = e^{\pi t}$, and
- explain how to find a particular solution if instead $h(t) = t^{2019}$.

Problem 2 Consider the dynamic programming problem

$$J_{t_0}(x) = \max_{u_t>0} \left\{ x_T + \ln x_T + \sum_{t=t_0}^{T-1} (u_t + \ln u_t) \right\}, \qquad x_{t+1} = x_t - u_t \qquad \text{starting at } x_{t_0} = x > 0.$$

It is possible to start at part (b) and deduce (a) afterwards.

(a) Calculate J_{T-1} and J_{T-2}.

(b) Use induction to show that for each $s = 0, 1, \ldots,$ we have $J_{T-s}(x) = x + C_{T-s} \cdot \ln \dfrac{x}{C_{T-s}}$ with $C_{T-s} > 0$ not depending on x.

(c) Consider the problem obtained in the limit $T \to +\infty$:

- State the associated Bellman equation.
- Why can we *not* expect the Bellman equation to have a (finite) solution $J(x)$? (*Hint:* Look at the limit of C_{T-s}.)

Problem 3 Let T, r and x_0 be constants, all > 0. Consider the variational problem

$$\min \int_0^T e^{-rt} \Big(x(t)^3 \cdot \dot{x}(t) \Big)^2 \, dt, \qquad x(0) = x_0, \quad x(T) = 2.$$

(a)
- State the associated Euler equation.
- State the conditions from the maximum principle, obtained by rewriting as an optimal control problem (maximization!) with control $u = \dot{x} \in (-\infty, +\infty)$.

Let from now on $x_0 = T = 1$ and $r = \ln 2$. Take for granted that this $x^*(t)$ is optimal:

$$x^*(t) = \sqrt{3 \cdot 2^t - 2} \qquad \text{so that} \quad \dot{x}^*(t) = \frac{3 \ln 2}{2} \cdot \frac{2^t}{x^*(t)} \qquad (*)$$

Hint: It is possible to answer the following part (b) *without* solving any differential equation, if you use formulae $(*)$. You are *not* asked to show or verify $(*)$.

(b)
- Calculate $p(1)$, where p is the adjoint variable from the maximum principle. (You are allowed to calculate the current-value adjoint $\lambda(1)$ instead.)
- Find an expression for how much, approximately, the *optimal value* changes if T increases from 1 to $1 + 1/144$. (If you did not manage to solve the previous bullet item, use the number e in place of $p(1)$ or of $\lambda(1)$.)

Problem 4 Let ϕ be the strictly increasing function $\phi(t) = t^{1/3} + e^{t/3}$, all $t \in \mathbb{R}$. Decide the quasiconcavity/quasiconvexity(/both/neither) of the three functions

$$f(z) = \phi(z+1) + \phi(z-1), \qquad g(x,y) = \phi(y - x \cdot (1-x)) \qquad \text{and} \qquad h(x,y) = \phi(x^{3/2} y^e),$$

f and g defined everywhere and h for $x \geq 0$, $y \geq 0$ (all domains convex).