ECON4140 Mathematics 3 exam 2020-05-29

- You are required to state reasons for all your answers.
  For the 2020 exam in particular:
  - Except where otherwise stated, all answers must be justified with calculations as if the exam were a school exam with textbook only;
  - As long as you are invoking theory known from the syllabus, you can reference it as it would have been expected in a school exam; and, where are asked to state particular instances of conditions or other theory, no citation is needed. This item relaxes the rules for referencing and citations; note however that *quoting from a source* is subject to normal citation practice (with quotations marks and references to the source).
    You are not expected to create any bibliography.
- You are permitted to use any information stated in an earlier letter-enumerated item (e.g. "(a)") to solve a later one (e.g. "(c)"), regardless of whether you managed to answer the former. A later item does not necessarily require answers from or information given in a previous one.
- "Suggested" weights: the grading committee is free to deviate. In particular, the committee might want to consider problems 2 and 3 together.

Problem 1 of 5. ***Suggested weight: 1/4*** For each real q, define the matrices

$$\mathbf{A}_q = \begin{pmatrix} q & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & q \end{pmatrix} \text{ and } \mathbf{B}_q = \begin{pmatrix} q & q^2 \\ q^3 & q^4 \\ q^5 & q^6 \end{pmatrix} \text{ and } \mathbf{C}_q = \left(\mathbf{A}_q \vdots \mathbf{B}_q\right) = \begin{pmatrix} q & 1 & 1 & q & q^2 \\ 1 & 1 & 1 & q^3 & q^4 \\ 1 & 1 & q & q^5 & q^6 \end{pmatrix}.$$

(a) For each real q, decide (i) the rank of $\mathbf{A}_q$, (ii) the rank of $\mathbf{C}_q$ (hint: do $\mathbf{A}_q$ first), and (iii) whether the the equation system $\mathbf{A}_q\mathbf{X} = \mathbf{B}_q$ has a solution. (You are not asked for uniqueness nor to solve.)

(b) (i) Show that $\mathbf{A}_q$ is never negative semidefinite, and decide when it is indefinite.

(ii) Complete the statement: «*From item* (i) *and the fact that not all eigenvalues of* $\mathbf{A}_q$ *are equal, it follows (for any* q*) that at least one eigenvalue is* [______].»

(c) Which of the vectors $(1, 0, -1)'$, $(-1, 0, 1)'$, $(q, 0, q)'$, is/are eigenvectors of $\mathbf{A}_q$? The answer could (but does not necessarily) depend on q.

(d) Let $q = -1$. $\lambda = 2$ is an eigenvalue for $\mathbf{A}_{-1}$. Find an associated eigenvector $\mathbf{v}$.

Problem 2 of 5. ***Suggested weight: 1/8*** Consider the dynamic programming problem $J_{t_0}(x_{t_0}) = \max_{u_t \in (-\infty,+\infty)} \Big\{ -x_T^2 + \sum_{t=t_0}^{T-1} (x_t - u_t^2) \Big\}$ subject to $x_{t+1} = x_t + u_t + w_{t+1}$, where $w_1,\ w_2, \ldots, w_T$ are given numbers.

- Show by induction that J_{T-s} has the form

$$J_{T-s}(x) = -\frac{1}{M_s}x^2 + 2B_s x + C_s \qquad \text{with } M_s > 0$$

  and where neither M_s, B_s nor C_s depend on x. (You are not asked to calculate these coefficients, but M_s will satisfy a fairly simple linear difference equation.)

If unable to do so, you can achieve partial score by instead calculating J_{T-1} and u^*_{T-2} for some special cases for the w's, e.g. $w_{T-1} = w_T = 0$. You will not get additional score for this if your answer to the above problem is at "B" level.

Problem 3 of 5. *Suggested weight: 1/4*

(a) Pick *one* of the following three bullet items and answer it.
If you submit answers to more, your best will count and the others be discarded.

- Find the general solution of $\ddot{w} = q \cdot (2\dot{w} - w + t)$ for $q = 0$, $q = 1/2$ and $q = 1$.
- Suppose that the differential equation system $\dot{\mathbf{z}} = \left(\begin{smallmatrix} 0 & b \\ c & 0 \end{smallmatrix}\right)\mathbf{z}$ (in the plane) has only one stationary state, namely $\mathbf{0}$; and, that the origin is not a saddle point.
Find the first coordinate z_1 of the general solution in terms of functions of a *real* variable.
- Find the general solution of the differential equation system for $\mathbf{z}$, under the same assumptions – also in terms of functions of a real variable.

(b) The nonlinear differential equation system

$$\dot{x}(t) = x \cdot (2 - x) - \max\left\{\frac{1}{y} - R\,,\,0\right\}, \qquad \dot{y}(t) = 2(x-1) \cdot y \qquad \text{(S)}$$

has a stationary state $(\tilde{x}, \tilde{y})$ with $\tilde{x} = 1$ and $\tilde{y} > 0$. Show that $(\tilde{x}, \tilde{y})$ is a saddle point.

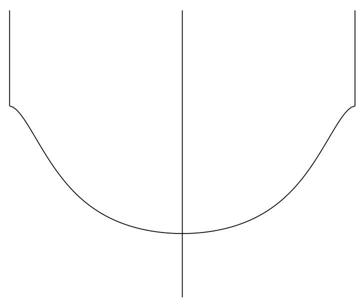
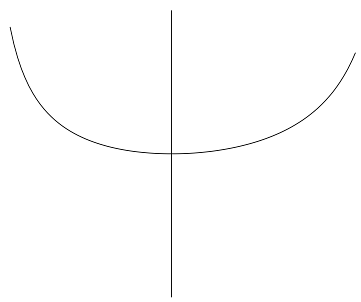
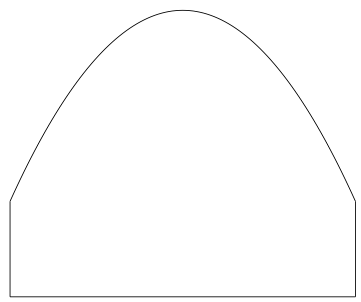
(c) Consider again system (S). In this question you shall find where in the phase plane there is vertical-only / horizontal-only motion.

(i) Deduce formulae for the nullclines (a.k.a. “null isoclines”) for the differential equation system; this deduction must be clear enough in absence of any plots;

(ii) Point out where any nullcline has a component that is a line segment. You can choose whether to indicate that in the formulae in (i) or in the sketch in (iii).

(iii) Indicate in a sketch where there is vertical resp. horizontal motion.

Possible hints for (iii) can be found in the below plots, but *beware that none are complete and all have inaccuracies.* Nevertheless one is accurate enough for (iii), once you add a missing element and then make those indications what (iii) asks for.

Problem 4 of 5. ***Suggested weight: 1/4*** Consider the optimal control problem

$$\max_{u(t)\geq 0} \int_0^T \ln(R+u)\,dt \quad \text{subject to} \quad \dot{x} = x\cdot(2-x) - u, \qquad x(0) = x_0, \qquad x(T) \geq 0$$

where the constants R, T and x_0 are all > 0. The problem is non-degenerate: you can *disregard* any possibility of the «p_0» constant being zero.

(a) (i) State the necessary conditions from the maximum principle.

(ii) Are these conditions also sufficient for this particular problem?
If not: is there any additional condition that would enable us to conclude that an admissible pair satisfying the necessary conditions from (i), will indeed solve the problem?

(b) Let $y(t) = p(t)$, the adjoint (costate) from the maximum principle. Show that the conditions from the maximum principle implies the differential equation system

$$\dot{x}(t) = x\cdot(2-x) - \max\big\{\frac{1}{y} - R\,,\,0\big\}, \qquad \dot{y}(t) = 2(x-1)\cdot y$$

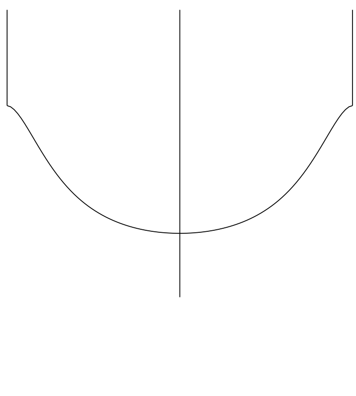
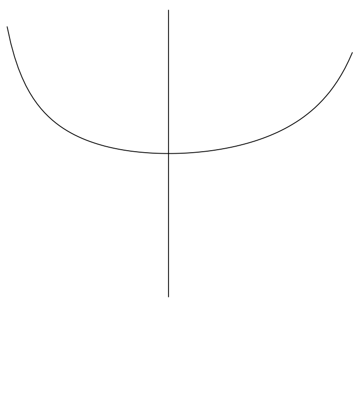
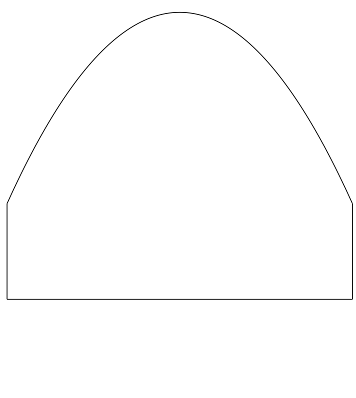
which is system (S) from Problem 3.

For part (c), use the fact (cf. Problem 3(b)) that system (S) has a saddle point $(\tilde{x},\tilde{y}) = (1,\tilde{y})$ with $\tilde{y} > 0$. Call the value of the optimal control problem $V(x_0)$, and consider the derivative V' at $x_0 = \tilde{x} = 1$ (taking for granted that $V'(1)$ exists).

(c) Show that we will always have $V'(1) \in [0,1]$ no matter what $R > 0$ and $T > 0$, and decide whether the following statement is true or false: *«when we vary $R > 0$ and $T > 0$, we can make $V'(1)$ attain any value in $(0,1)$.»*

(If you prefer the problem formulation «decide what values $V'(1)$ can attain and not, when $T > 0$ and $R > 0$ vary», then that is what is asked for – except you are not required to take a stand on whether 0 or 1 can be attained.)

In case you need the figures following Problem 3, they are repeated below:

Problem 5 of 5. ***Suggested weight: 1/8*** Let $\alpha < \beta$ be constants, $0 < \alpha < \beta < 1$. Define $f(x, y, z) = \left(\beta x^{\alpha} y^{\beta-\alpha} + \min\{\alpha y^{\beta}, z^{\beta}\}\right)^{1/\beta}$ whenever $x > 0$, $y > 0$ and $z > 0$.

(a) Show that f is quasiconcave.
(*Hint:* The problem can be solved without trying to calculate partial derivatives of f, and you might take note that f is not even C^1.)

(b) Is there any quick way to decide whether f is concave?

`<-- end of problem set -->`