ECON4140 Mathematics 3 exam 2020-06-23

- You are required to state reasons for all your answers.
  For the 2020 exam in particular:
  - Except where otherwise stated, all answers must be justified with calculations as if the exam were a school exam with textbook only;
  - As long as you are invoking theory known from the syllabus, you can reference it as it would have been expected in a school exam; and, where are asked to state particular instances of conditions or other theory, no citation is needed. This item relaxes the rules for referencing and citations; note however that *quoting from a source* is subject to normal citation practice (with quotations marks and references to the source).
    You are not expected to create any bibliography.
- You are permitted to use any information stated in an earlier letter-enumerated item (e.g. "(a)") to solve a later one (e.g. "(c)"), regardless of whether you managed to answer the former. A later item does not necessarily require answers from or information given in a previous one.
- "Suggested" weights: at the grading committee's discretion. The problem set was intended to *facilitate* weighting uniform over letter-enumerated parts.

⋆ *Note:* Problems 1–5 are of uneven length/workload with 5 having *four* letter parts, ending in a possibly difficult question.

Problem 1 of 5. Consider the following differential equation and difference equation:

$$\ddot{x} = \tfrac{1}{2}\dot{x} - \tfrac{1}{8}x + f(t) \qquad \text{(D)}$$

$$x_{t+2} = \tfrac{1}{2}x_{t+1} - \tfrac{1}{8}x_t + g_t \qquad \text{(E)}$$

(a) Find the general solutions of (D) and (E) when $f(t) = 2020 = g_t$ (constants).

(b) Explain how you would go forth to solve (D) when $f(t) = 2020+t^{2020}+2020\sin(2020\,t)$.

(c)
- Is (D) stable? Is (E) stable?
- Do we have tools to decide stability of (D) and/or (E) *without* solving?

Problem 2 of 5. Consider the differential equation system

$$\begin{aligned} \dot{x} &= y - \cos\frac{\pi x}{3} - \frac{1}{2} \\ \dot{y} &= |x| - y \end{aligned} \tag{S}$$

(a) The system has an equilibrium point (a.k.a. «stationary state») at $(x, y) = (1, 1)$. Show that it is a saddle point.

(b) The system has one more equilibrium point $(\tilde{x}, \tilde{y})$. Find and classify it:

- Decide if it is stable or not;
- Decide if it is oscillating or not;
- If unstable, decide if it is a saddle point.

(c) There is a non-constant particular solution $(x(t), y(t))$ which converges to $(1, 1)$ as $t \to +\infty$. Find $\lim\limits_{t\to+\infty} \dfrac{y(t) - 1}{x(t) - 1}$ for this solution path.

Problem 3 of 5. Consider the variational problem(s)

$$\max / \min \int_0^T e^{-rt} \ln\Big(G(x(t), \dot{x}(t))\Big)\, dt \quad \text{subject to} \quad x(0) = x_0, \quad x(T) = 0$$

where all constants are > 0. This problem concerns the function

$$G(x, y) = R + (2 - x) \cdot x - y, \qquad R > 0 \text{ constant}$$

but for part (a) it is probably a good idea – and it is worth partial score by itself – to first use a general G (but insert for $\partial G/\partial y = -1$).

(a) Write out the associated Euler equation.

(b) Suppose we have found an $x = x(t)$ that satisfies the Euler equation with $x(0) = x_0$ and $x(T) = 0$. Can we then conclude that this solves the maximization problem? The minimization problem? Neither?

Problem 4 of 5. Let $\beta \in (0,1)$ be a constant. Consider the dynamic programming problem

$$J_t(x_t) = \max_{u_t \in U} \sum_{s=t}^{T} \beta^{s-t}(x_t - \beta u_t^2) \qquad \text{subject to} \quad x_{t+1} = \beta x_t + (1-u_t)\cdot u_t$$

You can choose – once for the entire Problem 4 – whether to use $U = (-\infty, +\infty)$ or $[0,\infty)$ or $[0,1]$. Your choice might affect what arguments you need to complete part (a).

In the following, neither A_τ nor B_τ (nor a_τ nor b_τ) can depend on x, only on time τ *remaining* to the end of the planning horizon.

(a) Show by induction that the value $J_{T-\tau}(x)$ takes the form $A_\tau x + B_\tau$, and find a difference equation for A_τ.

If you prefer not to use the β^{s-t} formulation, you can equivalently show that the function $M_t = \beta^t J_t(x_t) = \max_{u_t \geq 0} \sum_{s=t}^{T} \beta^s (x_t - \beta u_t^2)$, takes the form $M_{T-\tau} = a_\tau x + b_\tau$; then find difference equations for a_τ and b_τ.

(b) Put $t = 0$. Let $T \to +\infty$ to get an infinite-horizon problem.

- State the associated Bellman equation for the value function $J(x)$.

  (In this problem you must use J.)

- Show that if $J(x) = Ax + B$ satisfies the Bellman equation (with A and B constants) then $A = 1/(1-\beta^2)$.

  (You are not asked to show that such a function actually solves the infinite horizon problem.)

Problem 5 of 5. Let $n \geq 2$ be a natural number and s be a real constant. Define $\mathbf{J}_n$ to be the $n \times n$ matrix where element (k, ℓ) is 1 if $k + \ell = n + 1$, and 0 otherwise[1]. Note that $\mathbf{J}_3$ has a «middle element» number $(2, 2)$, while $\mathbf{J}_4$ has no single middle element.

Let the matrix $\mathbf{A} = \mathbf{A}_{n,s}$ be defined as $\mathbf{I}_n + s\mathbf{J}_n$, where $\mathbf{I}_n$ is the $n \times n$ identity matrix.

(a) Let $\mathbf{a} = (1, 0, \ldots, 0)'$ and $\mathbf{b} = (0, \ldots, 0, 1)'$ have only a single nonzero element. Let $\mathbf{u} = \mathbf{a} + \mathbf{b}$ and $\mathbf{v} = \mathbf{a} - \mathbf{b}$.

For each n and each s: Check whether the vectors $\mathbf{u}$ and/or $\mathbf{v}$ is/are eigenvector for $\mathbf{J}_n$ and/or $\mathbf{A}_{n,s}$.

(b) In this part let $n = 3$ and $s \neq 0$.

- Calculate the characteristic polynomial of $\mathbf{A}$.
- Find – or disprove the existence of – an eigenvector $\mathbf{w}$ which is *not* a linear combination of $\mathbf{a}$ and $\mathbf{b}$.

(c) Suppose that the rank $\mathbf{A}_{n,1}$ equals n. Can then s be equal to 1? Your answer might depend on n.

(d) Let $\mathbf{Y}$ and $\mathbf{Z}$ be symmetric $n \times n$ matrices, both of which have 0 as their smallest eigenvalue. Prove the following facts about $\mathbf{M} = \mathbf{Y} + \mathbf{Z}$:

- $\mathbf{M}$ has no negative eigenvalues.
- [Might be difficult.] If $\mathbf{Y}$ and $\mathbf{Z}$ have *no eigenvector in common,* we know that $\mathbf{M}$ has all its eigenvalues positive.

`<-- end of problem set -->`

[1]Examples: $\mathbf{J}_2 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $\mathbf{J}_3 = \left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{smallmatrix}\right)$ while $\mathbf{J}_4 = \left(\begin{smallmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{smallmatrix}\right)$