University of Oslo / Department of Economics / NCF

ECON4140 Mathematics 3: the 2020-05-29 exam solved

- *Special considerations for the 2020 exam:*
  - The format is exceptional: 5hrs15mins with more tools available.
  - The committee must exercise considerable discretion. There is a risk that the format and the set together misses the usual level of difficulty. Therefore it is suggested that the committee attach more than usual weight to the official grade descriptions, and also consider the empirical grade distribution[1].
    This exam does not claim to be suited for any specific grading thresholds[2].
  - Therefore also this document is intended to be updated after the ordinary grading for the information of the appeals committee.[3]
  - The appeals committee must however be prepared to compare to a representative sample of the exam papers submitted.[4] One cannot expect/trust this note to guide the quantification of grades for individual papers or a low number of such.
- *Standard disclaimer:*
  - This note is not suited as a complete solution or as a template for an exam paper. It was written as guidance for the grading process – however, with additional notes and remarks for using the document in teaching later.
  - The document reflects what was expected in that particular semester, and which may not be applicable to future semesters.
- *Weighting:* Suggestions were stated in the problem set, rather than suggesting a usual (for this course) uniform over letters. Problem 1 has several short parts; the suggested weight for problem 2 might be more than the fraction of workload, and is intended to better reflect the weight on the topic in the course.

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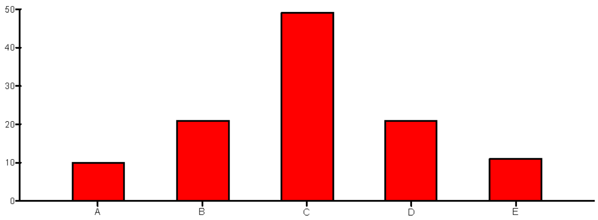
[1] According to a report from the Department's administration, covering the five years 2015-2019:



(112 passes; in addition seven failed)

Distribution: 10, 21, 49, 21, 11; that is cumulatively 9 %, 28 %, 71 % 90 %, 100 % over passes.

[2] which has defaulted to 91/75/55/45/40 percent in this course; the most recent four-hour Mathematics 2 exam did invoke Matematikkrådet's slightly tougher scale.

[3] *Added after grading:* Somewhat surprisingly, the ordinary grading thresholds were nearly applicable. A brief report from the grading is attached at the end, after the final page 14.

[4] *Added after grading:* This might not be as crucial as expected, cf. the previous footnote and the note summarizing the grading.

By mistake, the exam document indicates that the committee might want to consider 2 and 3 together. The appropriate parts are 3 and 4. This error was discovered the night before the exam, and was at that time considered too insignificant to be worth announcing a correction, possibly disturbing the workflow.

Problems/solutions/notes: The next pages will restate each problem as given, , each followed by a solution/annotations. Generally, what is in sans serif font in what is otherwise a solution, is a comment / an annotation; what says «Notes» in paragraph headings is of course notes as well.

The notes will sometimes refer to old exams, in particular to the «five full sets», those which have been assigned in its entirety as problems for the this semester's twelve «seminar» sets: 2017, 2016, 2012 and 2018 for seminars ##9–12 respectively, plus the 2015 partitioned over seminars ##5/10/12. Also a «mock exam» was produced afterwards – not so much a high-quality worked-through set (indeed, it had errors), but at least serving to adapt to the home exam format.

Problem 1 of 5. ***Suggested weight: 1/4*** For each real q, define the matrices

$$\mathbf{A}_q = \begin{pmatrix} q & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & q \end{pmatrix} \text{ and } \mathbf{B}_q = \begin{pmatrix} q & q^2 \\ q^3 & q^4 \\ q^5 & q^6 \end{pmatrix} \text{ and } \mathbf{C}_q = \left(\mathbf{A}_q \,\vdots\, \mathbf{B}_q\right) = \begin{pmatrix} q & 1 & 1 & q & q^2 \\ 1 & 1 & 1 & q^3 & q^4 \\ 1 & 1 & q & q^5 & q^6 \end{pmatrix}.$$

(a) For each real q, decide (i) the rank of $\mathbf{A}_q$, (ii) the rank of $\mathbf{C}_q$ (hint: do $\mathbf{A}_q$ first), and (iii) whether the the equation system $\mathbf{A}_q\mathbf{X} = \mathbf{B}_q$ has a solution. (You are not asked for uniqueness nor to solve.)

(b) (i) Show that $\mathbf{A}_q$ is never negative semidefinite, and decide when it is indefinite.

(ii) Complete the statement: «*From item* (i) *and the fact that not all eigenvalues of* $\mathbf{A}_q$ *are equal, it follows (for any* q*) that at least one eigenvalue is* [______].»

(c) Which of the vectors $(1, 0, -1)'$, $(-1, 0, 1)'$, $(q, 0, q)'$, is/are eigenvectors of $\mathbf{A}_q$? The answer could (but does not necessarily) depend on q.

(d) Let $q = -1$. $\lambda = 2$ is an eigenvalue for $\mathbf{A}_{-1}$. Find an associated eigenvector $\mathbf{v}$.

Solution:

(a) $\mathbf{A}_q$ has determinant $q\begin{vmatrix} 1 & 1 \\ 1 & q \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 1 \\ 1 & q \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = (q-1)^2$, so for $q \neq 1$, $\mathbf{A}_q$ has rank 3 and thus also $\mathbf{C}_q$ has rank 3. For $q = 1$, all elements are one and so all rows are proportional: there is only one linearly independent row, and so both ranks are 1 when $q = 1$. Solution exists iff $\mathbf{A}_q$ and $\mathbf{C}_q$ have the same rank, i.e. always.

(b) (i) Since element (2,2) is $1 > 0$, $\mathbf{A}_q$ cannot be negative semidefinite.
[Logically, this follows even without pointing out symmetry.]
The matrix is symmetric, and let us calculate the leading principal minors. Order 3: already done in (a) as $(q-1)^2$; order 2: $q-1$; order 1: q. All three are positive (implying positive definiteness) for $q > 1$. For $q < 1$: indefinite by the negativity of any even-order principal minor.
The answer is complete once we have established positive semidefiniteness for $q = 1$, which follows by continuity and taking limits as $q \to 1^+$. [This document chose that argument in order to indicate how brief the $q = 1$ case could be done. Alternatively: calculate the remaining four principal minors and note they are all nonnegative.]

(ii) «positive»: negating negative semidefiniteness $\Leftrightarrow$ negating «the largest eigenvalue is ≤ 0» $\Leftrightarrow$ some eigenvalue is positive.
[The fact that eigenvalues are nondistinct is redundant, but emphasizes that we can say more than «nonnegative». Still, «nonnegative» is a true statement, and the grading committee should take a stand on how to score it. It should get higher score than «real».]

(c) $\mathbf{A}_q\begin{pmatrix}1\\0\\-1\end{pmatrix} = \begin{pmatrix}q-1\\1-1\\1-q\end{pmatrix} = (q-1)\begin{pmatrix}1\\0\\-1\end{pmatrix}$, so this is an eigenvector (no matter what q).
The nonzero scaling $(-1,0,1)'$ is also an eigenvector (no matter what q).
$\begin{pmatrix}q\\0\\q\end{pmatrix}$ is an eigenvector iff $q \neq 0$ AND $\begin{pmatrix}1\\0\\1\end{pmatrix}$ is an eigenvector. $\mathbf{A}_q\begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}q+1\\1+1\\q+1\end{pmatrix}$, so:
Never an eigenvector because for the second element, $1+1$ is not a scaling of 0.

(d) $q = -1$, and we are *given* an eigenvalue of 2.

$$(\mathbf{A}_{-1} - \lambda\mathbf{I}) = \begin{pmatrix}-3 & 1 & 1\\ 1 & -1 & 1\\ 1 & 1 & -3\end{pmatrix} \sim \begin{pmatrix}0 & -2 & 4\\ 1 & -1 & 1\\ 0 & 2 & -4\end{pmatrix}$$

so the first equation is redundant and $v_2 = 2v_3$. Then $v_1 = v_2 - v_3 = v_3$. Putting $v_3 = 1$, we have an eigenvector $(1,2,1)'$. [Or any nonzero scaling will do – the problem deliberately only asked for «an» eigenvector.]

Notes: Several elements of Problem 1 resemble problem 1 of the 2012 exam, that was assigned (in full) for the second-to last seminar; even the q for the parameter and everything reducing to a matrix of ones when $q = 1$. Some remarks to each problem.

(a) Rank was an «earlier» topic, assigned for seminar #3, but also in exam 2015#1 for seminar #5 and exam 2017#4 for seminar #9, the aforementioned exam 2012#1 for seminar #11 and exam 2018#1 for seminar #12. The connection to linear equation systems in a subset of these though.

(b) Matrix definiteness is recurring; it suffices to mention the 2012 exam, but here there is a twist in the «Complete the statement» formulation.

(c) Verifying eigenvectors is very regularly given. The new thing – although given for the mock exam – is to check whether they are eigenvectors *or not*, when there is a parameter involved; where the $q(1,0,1)'$ making all sorts of problems to those who both are willing to divide by zero and not realizing the null vector is not an eigenvector.

(d) Finding an eigenvector given an eigenvalue – where the parameter is now fixed – should also be straightforward.

Problem 2 of 5. ***Suggested weight: 1/8*** Consider the dynamic programming problem $J_{t_0}(x_{t_0}) = \max_{u_t\in(-\infty,+\infty)} \Big\{ -x_T^2 + \sum_{t=t_0}^{T-1}(x_t - u_t^2)\Big\}$ subject to $x_{t+1} = x_t + u_t + w_{t+1}$, where $w_1,\ w_2, \dots, w_T$ are given numbers.

- Show by induction that J_{T-s} has the form

$$J_{T-s}(x) = -\frac{1}{M_s}x^2 + 2B_s x + C_s \qquad \text{with } M_s > 0$$

  and where neither M_s, B_s nor C_s depend on x. (You are not asked to calculate these coefficients, but M_s will satisfy a fairly simple linear difference equation.)

If unable to do so, you can achieve partial score by instead calculating J_{T-1} and u^*_{T-2} for some special cases for the w's, e.g. $w_{T-1} = w_T = 0$. You will not get additional score for this if your answer to the above problem is at "B" level.

Remarks first: The weight of this is deliberately heavier than the projected workload. This is to maintain the weight on this particular topic in particular, and more generally the early parts of the course (that were also given in physical lectures).

New this semester was to start it with the discrete time topics. How this would effect performance is unknown, and likely dwarfed by all the other exceptional turns of events. Anyway we did spend quite a bit of time on it, including applications to ECON4325. Though, the only trace of the «more advanced» part in this problem is, arguably, that in part (b) the b_t are arbitrary (remotely resembling a stochastic process view on it). However, dynamic programming was only part of two of the five full exam sets (as well as an utterly unsuccessful, due to my errors, part of the mock exam).

It was deliberate to have open control region (stationary max) in this problem, as Problem 4 has a possible endpoint maximum. Hence this, in some ways, straightforward problem (although there is a complicating element in involving unspecified constants); and, the «Plan B» problem deliberatly does not ask for J_{T-2} as the algebra after inserting the control does not add insight.

Solution: $J_T(x) = -x^2$, so it is OK for $s = 0$. Suppose OK at s. Then at $s+1$: $J_{T-(s+1)}(x) = \max_u \Big\{x - u^2 - \frac{1}{M_s}\cdot(x+u+w)^2 + 2B_s\cdot(x+u+w) + C_s\Big\}$ where w denotes w_{T-s}. Since the second-order coefficient $-(1+1/M_s)$ is $< -1 < 0$, the FOC $-2u^* - 2(x+u+w)/M_s + 2B_s = 0$ produces a maximum. Inserting

$$u^* = \frac{M_s B_s - (x+w)}{1+M_s} \tag{$\heartsuit$}$$

$$J_{T-(s+1)}(x) = x - \Big(\frac{M_sB_s - (x+w)}{1+M_s}\Big)^2 - \frac{1}{M_s}\Big(x + w + \frac{M_sB_s - (x+w)}{1+M_s}\Big)^2$$
$$+ 2B_s\Big[x + w + \frac{M_sB_s - (x+w)}{1+M_s}\Big] + C_s \qquad (\spadesuit)$$

This is indeed quadratic in x, and we only need to point out that because of the negative signs in front of the big parentheses, *the second-order coefficient becomes negative.* [*Once this is pointed out,* it likely goes without saying that as $-1/M_{s+1}$ will be negative, then M_{s+1} will have the right sign. It is not necessary to calculate it, but if one does, $\frac{1}{M_{s+1}}$ becomes $\frac{1}{(1+M_s)^2} + \frac{1}{M_s}\frac{M_s^2}{(1+M_s)^2} = \frac{1+M_s}{(1+M_s)^2}$ which indeed yields $M_{s+1} = 1+M_s$ («fairly simple») and so $M_s = s+1$.]

For those who go for the «plan B» option here, the calculations for J_{T-1} will be about as ($\spadesuit$) with $B_0 = C_0 = 0$ and $M_0 = 1$ and likely choosing $w = 0$: that yields $x - (x/2)^2 - (x/2)^2 + 0 + 0$, i.e. $J_{T-1}(x) = x - \frac{1}{2}x^2$. Then at $T-2$, insert $M_1 = 2$, $M_1B_1 = 1$ and $C_1 = 0$ into ($\heartsuit$); presuming that also here the $w = 0$ particular case is chosen, we obtain $u^*_{T-2} = (1-x)/2$ where $x = x_{T-2}$.

Problem 3 of 5. ***Suggested weight: 1/4***

(a) Pick *one* of the following three bullet items and answer it.
If you submit answers to more, your best will count and the others be discarded.

- Find the general solution of $\ddot{w} = q \cdot (2\dot{w} - w + t)$ for $q = 0$, $q = 1/2$ and $q = 1$.
- Suppose that the differential equation system $\dot{\mathbf{z}} = \left(\begin{smallmatrix} 0 & b \\ c & 0 \end{smallmatrix}\right)\mathbf{z}$ (in the plane) has only one stationary state, namely $\mathbf{0}$; and, that the origin is not a saddle point.
Find the first coordinate z_1 of the general solution in terms of functions of a *real* variable.
- Find the general solution of the differential equation system for $\mathbf{z}$, under the same assumptions – also in terms of functions of a real variable.

(b) The nonlinear differential equation system

$$\dot{x}(t) = x \cdot (2 - x) - \max\left\{\frac{1}{y} - R\,,\, 0\right\}, \qquad \dot{y}(t) = 2(x-1) \cdot y \tag{S}$$

has a stationary state $(\tilde{x}, \tilde{y})$ with $\tilde{x} = 1$ and $\tilde{y} > 0$. Show that $(\tilde{x}, \tilde{y})$ is a saddle point.

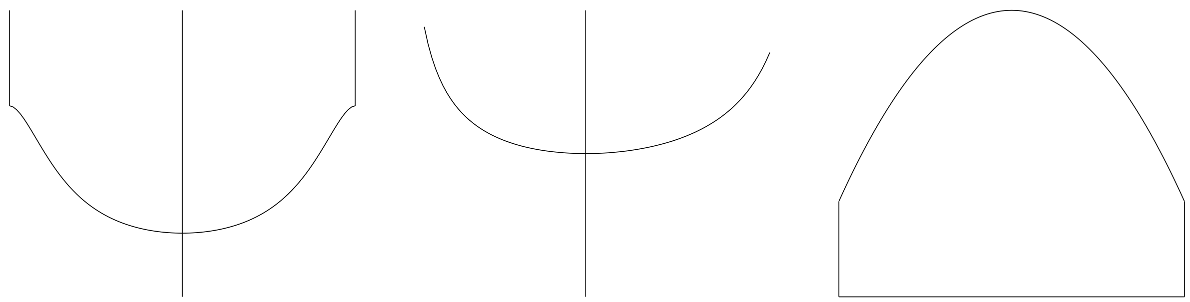
(c) Consider again system (S). In this question you shall find where in the phase plane there is vertical-only / horizontal-only motion.

(i) Deduce formulae for the nullclines (a.k.a. "null isoclines") for the differential equation system; this deduction must be clear enough in absence of any plots;

(ii) Point out where any nullcline has a component that is a line segment. You can choose whether to indicate that in the formulae in (i) or in the sketch in (iii).

(iii) Indicate in a sketch where there is vertical resp. horizontal motion.

Possible hints for (iii) can be found in the below plots, but *beware that none are complete and all have inaccuracies.* Nevertheless one is accurate enough for (iii), once you add a missing element and then make those indications what (iii) asks for.



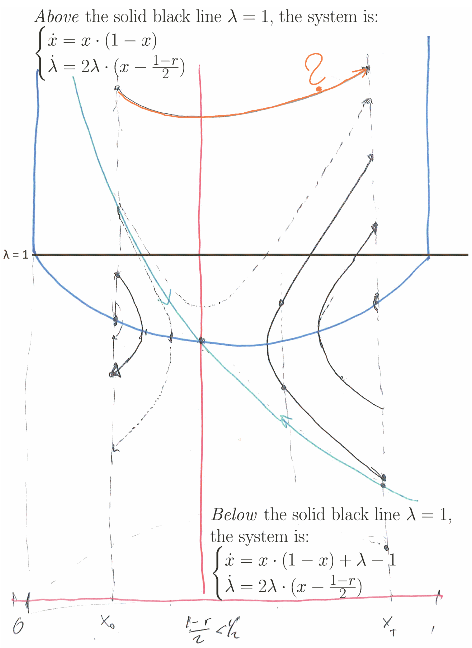
Notes on Problem 3 vs seminar problems etc.

(a) It has been communicated that we require a real solution, and possibly that makes it harder to get an answer out by just feeding it into a CAS. The first item with $q = 0$ is arguably easy, if one hasn't forgotten the basics at home.

Giving *options* like this is not common. There is a possible issue with the formulation of the second bullet item, in that it is not the most familiar. The grading committee should take note that linear autonomous differential equation systems typically score well on exams, and this is not intended to be hard. If scores indicate that the wording might have been confusing, the committee might consider a low weight on part (a). It was deliberate not to ask for the full solution as dividing by b would likely be a bit of work trying not to get a sign wrong.

(b) Classification for nonlinear systems was covered in seminar #8, but the optimal control part of the course spent quite a lot of effort on the system $\dot{x} = x \cdot (1-x) - \max\{1-\lambda, 0\}$ and $\dot{\lambda} = \lambda \cdot [r - 1 + 2x]$. For example, one problem from seminar #12 was to extract properties from the sketch given below (that had appeared in a video lecture). Also, the mock exam had an arguably nastier one, where one were also asked to identify the (asymptotics of the) convergent saddle path.

(c) Part (c) is not unlike a problem given in the mock exam, and the system is likely even closer to one that was stressed in the lectures and assigned for a seminar:



(Picture that was part of seminar assignment 12 – that one with time-discounting.)

This has several qualitative properties in common with the leftmost of the diagrams given in the exam problem set, which is the qualitatively «useful» one. Note, that one does not give the precise nullclines of the actual system; Part of the problem is to distil what is essential for qualitative analysis, and the big error – the inflections close to $x = 0$ and $x = 2$ – is not. If the candidates want to make a computer assisted sketches for themselves, they are free to do so – but anyone who spends exam time tweaking the parameter to replicate the exact plots, has missed target and has even been warned.

Need for clarification, parts (b) and (c): Problems 3 and 4 were constructed together, but a «Let $R > 0$ be a constant.» qualification (like in Problem 4) was forgotten from the beginning of part (b). This was noticed when it was asked during the exam whether it was intended not to give a specific value for R (which it was). It is arguably natural to read the problem as to take for granted that such a stationary state exists, and then the calculations do however only depend on $\tilde{y}$ being > 0 as stated. The corresponding problems from the seminar and mock exam did consider what happened in the first quadrant, and likely those who even spot the possible issue will also resolve it – but the committee should be cautious to check for adverse consequences.

Part (c) has an unintended consequence: A nullcline for y is apparently where $2(x-1)y = 0$, i.e. on one of two straight lines; however the formula for $\dot{x}$ is not well-defined when $y = 0$. In the course it has been stressed to not divide by zero. Candidates might even be justified in thinking that the $2(x-1)y$ form is a trap for those who *omit* $y = 0$ (and indeed, that was the intention of an earlier draft, and overlooked when it was finalized). It might be more just to simply disregard whether $y = 0$ was included or overlooked – and also to allow the interpretation that y is restricted to being > 0. On the other hand, the vertical half-lines at $x = 0$ and $x = 2$ should be much more expected from the course, cf. the above sketch from seminar 12.

Solution:

(a) [Two alternatives covered]

- $q = 0$: The equation $\ddot{w} = 0$ has general solution $\underline{\underline{Kt + L}}$.

  For the other cases, there is a particular solution $w^* = Mt + P$ satisfying $0 = 2\dot{w}^* - w^* + t = 2M - (Mt + P) + t$ so $M = 1$ and $P = 2$, and characteristic roots solving $r^2 - 2qr + q = 0$, so that $r = q \pm \sqrt{q^2 - q}$.

  $q = \frac{1}{2}$ gives non-real roots $r = \frac{1}{2} \pm \frac{1}{2}i$. Solution $\underline{\underline{2 + t + e^{t/2}\big(C \cos \frac{t}{2} + D \sin \frac{t}{2}\big)}}$.

  $q = 1$ gives the double root $r = 1$ and solution $\underline{\underline{2 + t + e^{t}(A + Bt)}}$.

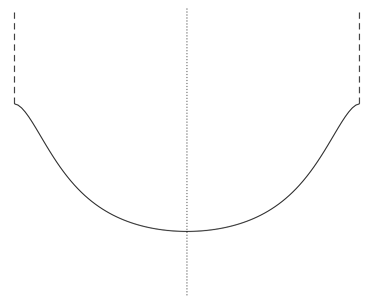
- For the stationary state to be unique, the coefficient matrix must be invertible, and so its determinant $-bc$ is nonzero. Furthermore, the determinant cannot be negative, as the origin is not a saddle point. So $bc < 0$. As the trace is zero, the eigenvalues (characteristic roots) are then $\pm\sqrt{bc}$ which are purely imaginary. Therefore, for the first coordinate we have $\underline{\underline{z_1(t) = A\sin(t\sqrt{|bc|}) + B\cos(t\sqrt{|bc|})}}$.

- [Hopefully, no-one will choose this additional work; the option was included just in case someone would find the second bullet item unclear. Solve $bz_2 = \dot{z}_1$ and beware that $b^{-1}\sqrt{|bc|} = \sqrt{|c/b|}\,\mathrm{sign}\, b$.]

(b) We have a saddle point if the Jacobian has negative determinant at the stationary state. Jacobian: $\begin{pmatrix} 2-2x & y^{-2} \\ 2y & 2(x-1) \end{pmatrix}$, which reduces to $\begin{pmatrix} 0 & \tilde{y}^{-2} \\ 2\tilde{y} & 0 \end{pmatrix}$ at $(\tilde{x}, \tilde{y})$; the determinant becomes $= -2/\tilde{y}$ which is < 0 as we are given that $\tilde{y} > 0$.

(c) For $\dot{x} = 0$: The solid curves in the diagram, have only vertical motion. That is iff $\max\{\frac{1}{y} - R, 0\} = x(2-x)$. For $x \in (0,2)$: the nonlinear part, the curve $y = \big(R + x(2-x)\big)^{-1}$.
Line segments: At $x = 0$ or $x = 2$ there are vertical line segments (dashed below) for $1/y - R \leq 0$ i.e. $y \geq 1/R$.

For $\dot{y}$: $2(x-1)y = 0$ iff $x = 1$ or $y = 0$, but the system is not well-defined at $y = 0$. A straight line $x = 1$ (dotted below), where there is only horizontal motion.



[This copies the relevant plot from the hint, only with the vertical part of the x-nullcline dashed, and the y-nullcline dotted. Inaccuracies kept. As the indication of horizontal/vertical motion is given in words above this sketch, arrows are arguably optional if the rest is good enough.]

Problem 4 of 5. ***Suggested weight: 1/4*** Consider the optimal control problem

$$\max_{u(t)\geq 0} \int_0^T \ln(R+u)\,dt \quad \text{subject to} \quad \dot{x} = x\cdot(2-x) - u, \qquad x(0) = x_0, \qquad x(T) \geq 0$$

where the constants R, T and x_0 are all > 0. The problem is non-degenerate: you can *disregard* any possibility of the «p_0» constant being zero.

(a) (i) State the necessary conditions from the maximum principle.

(ii) Are these conditions also sufficient for this particular problem?
If not: is there any additional condition that would enable us to conclude that an admissible pair satisfying the necessary conditions from (i), will indeed solve the problem?

(b) Let $y(t) = p(t)$, the adjoint (costate) from the maximum principle. Show that the conditions from the maximum principle implies the differential equation system

$$\dot{x}(t) = x\cdot(2-x) - \max\left\{\frac{1}{y} - R\,,\,0\right\}, \qquad \dot{y}(t) = 2(x-1)\cdot y$$

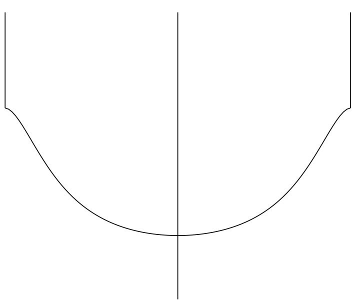
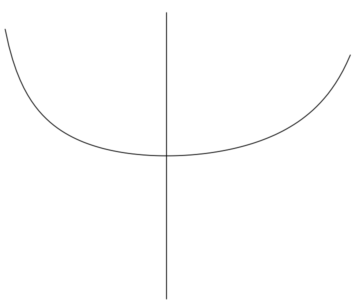
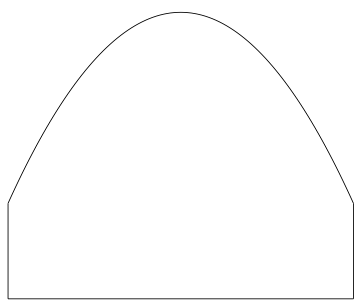
which is system (S) from Problem 3.

For part (c), use the fact (cf. Problem 3(b)) that system (S) has a saddle point $(\tilde{x},\tilde{y}) = (1,\tilde{y})$ with $\tilde{y} > 0$. Call the value of the optimal control problem $V(x_0)$, and consider the derivative V' at $x_0 = \tilde{x} = 1$ (taking for granted that $V'(1)$ exists).

(c) Show that we will always have $V'(1) \in [0,1]$ no matter what $R > 0$ and $T > 0$, and decide whether the following statement is true or false: *«when we vary $R > 0$ and $T > 0$, we can make $V'(1)$ attain any value in $(0,1)$.»*

(If you prefer the problem formulation «decide what values $V'(1)$ can attain and not, when $T > 0$ and $R > 0$ vary», then that is what is asked for – except you are not required to take a stand on whether 0 or 1 can be attained.)

In case you need the figures following Problem 3, they are repeated below:

Solution first (please mind the notes to follow below):

(a) [Omitted «t,» from H because there is no explicit t.]
With $H(x, u, p) = \ln(R + u) + px(2 - x) - pu$, the conditions are as follows:

> $*$ The optimal u^* maximizes $\ln(R + u) + px(2 - x) - pu$ over $u \geq 0$.
>
> $*$ $\dot{p} = -p \cdot (2 - 2x)$ with $p(T) \geq 0$, and $p(T) = 0$ if $x^*(T) > 0$.

Without further knowledge on p, we do not know whether $p \cdot x(2-x)$ is concave in x, so we cannot apply Arrow (nor Mangasarian). But if we can establish $p(t) \geq 0$, either of these will apply.

(b) $\dot{y} = -2y(1 - x)$ from the differential equation for p, so we are done when we can establish the «max» term in the diff.eq. for x. From the maximum principle, u^* maximizes $\ln(R + u) - pu$, which is concave in u, and the maximizer is the stationary point (where $1/(R + u) = p$ i.e. $u = 1/p - R$) iff positive and 0 otherwise; that is, $u^* = \max\{1/p - R, 0\}$. Inserting y for p and insert this for u in $\dot{x} = x(2 - x) - u$, and we are done.

(c) $V'(x_0) = p(0) = y(0)$. For us to end up at $x(T) = 0$ or $y(T) = 0$ when starting at $x_0 = \tilde{x}$, we must start out *below* the stationary state in the phase diagram, hence $y(0) \in (0, \tilde{y})$: near 0 when $T \approx 0$ (for then V is small), and near the saddle point as $T \to +\infty$ (for then we must spend long time). And $\tilde{y} = 1/(R + 1)$ can get $\to 1$ by letting $R \to 0$. So the answer is the interval $\underline{\underline{(0, 1)}}$.

Notes and remarks: As mentioned in the notes for Problem 3, this is a modification of an optimal extraction problem given in a lecture and elaborated on in seminar problems; that one had discounting but quadratic (non-monotonous!) running income, and so this exam problem should not be any harder despite the R parameter. In particular, the absence of discounting means there is no distinction between present-value and current-value formulation, which should ease part (b).

In particular for the individual parts:

(a) One is free to include admissibility in the conditions from the maximum principle – and if one does, it is perfectly fine to write «without stars» as well, much like one usually would do in a Lagrange problem in Mathematics 1 or 2.

Note, the first condition is maximality and not the FOC (and that will materialize in (b)); it was deliberate to give a problem which will in some cases give endpoint maximum. It *is* however perfectly fine to say that u^* maximizes $\ln(R + u) - pu$

(omitting the terms w/o u) over $u \geq 0$, and it is even OK to write that it implies $1/(R + u^*) = p$ if that gives $u^* > 0$ and $u^* = 0$ otherwise.

As for sufficiency, it «should be clear» to an economist that higher stock is good – but that is not required for (a), as it requires the knowledge that p is a shadow price. Likely, there will be answers claiming sufficiency; hopefully they will actually justify $p \geq 0$ (e.g. by invoking that interpretation), it is doubtful that this goes without saying at this level, but the committee might consider the full answer to Problem 4 in judging whether the candidate must have had good reason to take it as obvious.

(b) With the «max» in the given formula, there is no excuse being unable to handle endpoint maximum.

(c) The most crucial part of (c) is that the adjoint is the shadow price is the derivative of the value function. One might consider awarding a passing grade on (c) for merely pointing it out, but one might consider to assess whether a partial answer shows any understanding of it elsewhere, like in discussing the nonnegativity of shadow prices for sufficiency.

It isn't completely obvious that the shadow price must be upper bounded by $\tilde{y}$, but from the problem text it isn't hard to guess (one is asked to show an upper bound, and if forced to make a wild guess, the point in the diagram would be an obvious thing to point at). Arguing *why* is likely to take a good paper; arguing that we could get $V'(1)$ close to zero as the value vanishes as planning horizon $\searrow 0$ isn't farfetched but not trivial, but arguing that we can get $V'(1) \nearrow 1$ by letting T *and* R grow, could very well be the hardest element of the exam.

Problem 5 of 5. ***Suggested weight: 1/8*** Let $\alpha < \beta$ be constants, $0 < \alpha < \beta < 1$. Define $f(x, y, z) = \Big(\beta x^\alpha y^{\beta-\alpha} + \min\{\alpha y^\beta, z^\beta\}\Big)^{1/\beta}$ whenever $x > 0$, $y > 0$ and $z > 0$.

(a) Show that f is quasiconcave.
(*Hint:* The problem can be solved without trying to calculate partial derivatives of f, and you might take note that f is not even C^1.)

(b) Is there any quick way to decide whether f is concave?

Notes first: For the last seminar, one determined concavity of a CES this way. Here we have a nested CES form (harder) with the degree of homogeneity being precisely one (simplifying). This method was arguably stressed more this year than usual.

The following proposed solution does not mention that the open first orthant is a convex set, as I think that can be omitted; also I think one is allowed to know the concavity of the bivariate Cobb–Douglas for these parameters, nor do I make an argument for the positivity of the function. Maybe a one-liner for homogeneity would be in place, or maybe – at the end of the exam set – those who can actually identify the relevant piece of theory, will have answered part (b) fully; I suggest to be cautious about papers who write e.g. "check the degree of homogeneity" as if all quasiconcave functions were necessarily homogeneous; on the other hand, any degree of homogeneity being $\in (0, 1]$ would do.

Solution:

(a) Because the constants are $\in (0, 1)$, αy^β and z^β are concave functions, and so is the minimum of two concave functions. Also, $\beta x^\alpha y^{\beta-\alpha}$ is a concave Cobb–Douglas, as the exponents sum to $\beta < 1$. The sum of these concaves is concave. Thus f is an increasing transformation $t^{1/\beta}$ of a concave function, hence quasiconcave.
[Note, the sum argument uses concavity – the sum of quasiconcaves need not be quasiconcave.]

(b) Yes: for a positive quasiconcave function, *homogeneity of degree* $\in (0, 1]$ implies concavity, and this is homogeneous of degree 1.

Om sensuren i ECON4140 våren 2020

Sensorveiledningen maner til å vise omhyggelighet i å sette karakter(grens)er. Det viser seg at oppgavesettet jevnt over produserte en god differensiering mellom sterkere og svakere kandidater, og at de vanlige karaktergrensen var svært nær å kunne anvendes.
Svakeste "A" scorte 89 prosent. En B-grense på 75 pst befant seg i et relativt vidt intervall mellom beste C og svakeste B. Å beholde C-grensen på 55 pst gav også et synlig skille mellom svakeste C og beste D. E-intervallet i standardskalaen er smalt, men de E-ene som ble tildelt, var merkbart svakere enn D-ene og vesentlig bedre enn hva vi strøk.
Dette produserte også en nær symmetrisk fordeling over beståtte prøver, med like mange A-er som E-er og én B flere enn D. Også gjennomsnittlig prosentscore - over beståtte - var det som normalt ville ha vært en middels C.

Oppgavesettet hadde mer enn én nøtt: både 4c og 5b scoret i gjennomsnitt til stryk. De vanskeligere delene var dog bedre besvart enn i hva en av oss hadde i lagret fra tidligere års sensur. Det var få tegn på tidsnød; tvert imot var det mange som forsøkte både induksjonsbeviset "plan B"-alternativet i oppgave 2. (Derimot var det ingen som førskte flere alternativer på 3(a), selv om det var overraskende mange merkelige svar på denne delen.)
En (ikke for vanskelig) oppgave ekstra ville neppe ha gjort skade.

Vi gir noen kommentarer om de enkelte oppgavene.

1. Oppgave 1 hadde bedre score enn gjennomsnittet.
En vanlig feil i del (b) var å bare sjekke hoveddiagonalen - altså, som om alle kryssledd var utelatt. En tilsvarende feil var gjenganger i 4(a), tilstrekkelige betingelser, der mange tror konkavitet i x og konkavitet i u gir konkavitet i (x,u). At kryssledd spiller en rolle burde være kjent siden lenge før man begynte på Matematikk 2.
Oppgave 1 hadde for øvrig en god del ubegrunnete delsvar som tyder på at det ble brukt verktøy som gir svaret ut uten begrunnelse. Dette er på ingen måte forbudt, men det er heller ikke fullstendig svar på et oppgavesett som klargjør at "all answers must be justified with calculations as if the exam were a school exam with textbook only". På den andre siden er slike ufullstendige svar heller ikke verdiløse, særlig ikke når man må forstå spørsmålet for å finne svaret med verktøy.

2. Oppgave 2 scoret litt under gjennomsnittet, og full score var sjelden vare.
En vanlig feil var å ikke gjenkjenne en kvadratisk funksjon, og å ikke gjøre noe forsøk på å peke på at andreordenskoeffisienten ville få riktig fortegn. Mange forsøkte å regne helt ut og finne konstanter de ikke var bedt om, men når man er så overtydelig, så bør man i det minste gjøre noe for å besvare det det spørres etter.

3. Besvart omtrent like godt som oppgave 1, men mye mer ujevnt.
3(b) var den best besvarte delen av settet, med et gjennomsnitt på A-nivå (!).
Del (a), derimot; det går tydeligvis an å komme seg gjennom dette kurset uten å kunne integrere w''=0 to ganger for å få w=Ct+D, og uten å skjønne hva en "partikulær" løsning er for noe. Gjennomgående ble første alternativ på del (a) valgt.
3(c) var besvart rundt D-nivå. Fasediagrammer er tydeligvis en vanskelig del av kurset, selv om de ved minst ett annet lærested gis på Matematikk 2-nivå.

4. 4(a) og (b) var besvart omtrent som på snittet i settet. (c) under beståttgrensen.
Å stille opp betingelser pleier å være en freebie, men her var det inkludert spørsmål om tilstrekkelighet, som gir muligheten til å feile det elementære spørsmålet om hva konkavitet i to variabler krever; det er rett nok en sum av univariat konkave, men for mange skriver at siden H''xx og H''uu begge er negative, så ... . Og ut fra spørsmålsformuleringen i prøveeksamen er det overraskende hvor mange som ikke engang bryr seg om fortegnet på det kvadratiske leddet før de kaller det kvadratisk.
4 (c) var vanskelig. Flere hadde fått med seg at sadelpunktet spiller en rolle, men få klarte å argumentere for at man måtte være nedenfor det.

5. Svakest score til tross for tidvis generøs bedømmelse.
Denne oppgaven hadde svært mange feilaktige påstander. I 5(a) brukte man ofte at uttrykket i parentes er en sum av kvasikonkave. Det impliserer ikke kvasikonkavitet. Flere forsøkte å påstå at å opphøye en konkav i en konstant (som her er >1) gir en konkav.
Enkelte kandidater viste til bokas resultater om CES-funksjoner. Selv om de ikke helt kommer til anvendelse, gav dette delscore; gjenkjennelse er tross alt et vesentlig element i en åpen bok-eksamen, og her er nettopp det å nøste to grensetilfeller i CES-familien som driver resultatet.

Sensorene