UNIVERSITY OF OSLO
DEPARTMENT OF ECONOMICS

Exam: ECON4140 - Mathematics 3: Differential Equations, Static and Dynamic Optimization

Date of exam: 26 May, 2021 **Grades are given:** 16 June 2021

Time for exam: 09.00 – 12.00

The problem set covers 3 pages (incl. cover sheet)

Resources allowed:

- All written and printed resources – as well as two alternative calculators - are allowed

The grades given: A-F, with A as the best and E as the weakest passing grade. F is fail.

University of Oslo / Department of Economics **English version only**

ECON4140 Mathematics 3: exam 2020-05-26, 09:00-12:00

There are 2 pages of problems to be solved.
All printed and written material may be used, as well as both the approved calculators.
Grades given run from A (best) to E for passes, and F for fail.

- You are required to state reasons for all your answers.
- You are permitted to use any information stated in an earlier letter-enumerated item (e.g. "(a)") to solve a later one (e.g. "(c)"), regardless of whether you managed to answer the former. A later item does not necessarily require answers from or information given in a previous one.

Problem 1 ***Suggested weight:*** *30 %.*
Let q be an arbitrary real constant. For each such q, let $\mathbf{A} = \begin{pmatrix} 2 & 1 & 0 \\ 4 & -1 & 0 \\ 0 & 0 & e^q \end{pmatrix}$.

(a) Decide whether claim «$\mathbf{u} = (0, 0, q)'$ is an eigenvector of $\mathbf{A}$» is:

- ☐ Always true? (I.e. true for all real q.)
- ☐ Always false? (I.e. false for all real q.)
- ☐ True for at least one q and false for at least one q?

(b) $\lambda = -2$ is an eigenvalue for $\mathbf{A}$. Calculate an associated eigenvector $\mathbf{v}$.

(c) In this question you shall find the last eigenvalue of $\mathbf{A}$: an eigenvalue γ that does not depend on q and does not equal λ:

- For full score: calculate γ without calculating any determinant.
- For score up to a «C»: calculate γ by means of the characteristic polynomial.

(d) $(0, 0)$ is an equilibrium point (a stationary state) for the differential equation system

$$\dot{x} = 2x + y$$
$$\dot{y} = 4 \sin x - y$$

There are indeed non-constant solution paths $(x(t), y(t))$ that converge to $(0, 0)$ as $t \to +\infty$. Find $\lim_{t\to+\infty} \frac{y(t)}{x(t)}$ for such a path.
(*Hint:* the solution is connected to either (a) or (b) or (c) above.)

Problem 2 ***Suggested weight:*** *20 %.*

(a) Find the *general* solution of the difference equation $x_{t+2} - 2x_{t+1} + \frac{5}{4}x_t = 20$.

(b) Find the *general* solution of the differential equation $\ddot{x} - 2\dot{x} + \frac{5}{4}x = 20$.

(c) *Explain how to* find a particular solution of $\ddot{x} - 2\dot{x} + \frac{5}{4}x = 20 + \cos\frac{t}{\pi}$.

Problem 3 ***Suggested weight:*** *20 %.*
Let $Q \in (0, 1]$ be a constant. Consider the dynamic programming problem

$$J_{t_0}(x) = \max_{u_t \in (0,Q]} \left\{ \tfrac{1}{Q} \ln x_T + \sum_{t=t_0}^{T-1} \ln(u_t x_t) \right\}, \qquad x_{t+1} = (1 - u_t^2) \cdot x_t, \qquad x_{t_0} = x > 0$$

General hints: You might need that $\ln((1-u^2)x) = \ln(1+u) + \ln(1-u) + \ln x$. And, at some stage, sketching a graph might be useful. It is possible to start at part (b).

(a) Let in this part $Q = 1$. Calculate J_{T-1} and u^*_{T-2}. (You need not calculate J_{T-2}.)

(b) Use induction to show that we have $J_{T-\tau}(x)$ of the form $A_\tau + B_\tau \ln x$ and where neither A_τ nor B_τ nor the optimal $u^*_{T-\tau}$ depend on x. (*Hint:* Make one more assumption.)

Part (b) does not ask you to calculate out A_τ nor B_τ nor $u^*_{T-\tau}$, but recall that Q could be an arbitrary constant in $(0, 1]$.

Problem 4 ***Suggested weight:*** *30 %.*
Let $s \approx 0$ be a constant. Consider – but do not try to solve! – the optimal control problem

$$\max_{u(t) \geq 0} \int_s^3 \left(ue^{t-x} + \sin(tx) \right) dt \qquad \text{subject to} \quad \dot{x} = x - 2 - u, \quad x(h) = 2, \quad x(3) \geq 1$$

(a) State the conditions from the maximum principle.

(b) Consider the (admissible!) control $u(t) = 0$ for all t. Show that this cannot be optimal. (*Hint:* Proof by contradiction might be an idea.)

(c) Suppose a solution exists. The optimal value depends on initial time s, call it $v(s)$. Show that $v'(0) = 0$.

There are several terms that zero out, and for full score, you must justify each of them correctly. You cannot assume that $u^* > 0$ everywhere even if we know that u^* must somewhere be > 0.