ECON4140 Mathematics 3: *home exam* 2021-08-10

- You are required to state reasons for all your answers. For this 2021 exam:
  - Except where otherwise stated, all answers must be justified with calculations as if the exam were a school exam with textbook only;
  - As long as you are invoking theory known from the syllabus, you can reference it as it would have been expected in a school exam; and, where are asked to state particular instances of conditions or other theory, no citation is needed. This item relaxes the rules for referencing and citations; note however that *quoting from a source* is subject to normal citation practice (with quotations marks and references to the source).
    You are not expected to create any bibliography.
- You are permitted to use any information stated in an earlier letter-enumerated item (e.g. "(a)") to solve a later one (e.g. "(c)"), regardless of whether you managed to answer the former. A later item does not necessarily require answers from or information given in a previous one.
- Grades given run from A (best) to E for passes, and F for fail.
  Suggested weights: equal weighting over letter-enumerated parts. The grading committee is free to deviate.

Two problem pages, starting next page.

Please submit one single PDF file.

Problem 1. Let $a > 0 \geq b$ be constants, and let $\mathbf{A} = \begin{pmatrix} a & b \\ a & b \end{pmatrix}$.

(a) Is $\mathbf{u} = (-b,\ a)'$ an eigenvector for $\mathbf{A}$? (Note that $\mathbf{u} \neq \mathbf{0}$, as $a > 0$.)

(b) Find an eigenvector associated to the eigenvalue $\lambda = a + b$.

(c) Note how λ equals the ratio $\rho = \dfrac{\text{sum of elements}}{\text{number of rows}}$. Decide which statement is true:

- ☐ Every $n \times n$ matrix has an eigenvalue that equals this ratio ρ.
- ☐ Every $n \times n$ matrix has an eigenvalue with real part equal to that ratio ρ, even if a real eigenvalue may fail to exist when n is even.
- ☐ The previous statement is true for $n = 2$ (and $n = 1$!), but false for $n \geq 3$.
- ☐ No such rule for $n > 1$; such an eigenvalue is a coincidence.

Consider from now on the the differential equation system (D) given by

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \mathbf{A} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} t + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \sin(e\pi t) \qquad \text{(D)}$$

(d) *Without* solving: is it possible from parts (a), (b) and (c) to know whether (D) is globally asymptotically stable? (The answer might depend on a and b.)

(e) Consider again system (D):

- *Deduce* – write out the calculations in detail! – a second-order differential equation for y (note, for «y» and not x);
- *Explain how to* find the general solution of that differential eq. when $a > |b|$.

Problem 2. Let $\mathbf{B}$ be a matrix of order $m \times n$, where $m < n$.

(a) In part (a), suppose that the rank of $\mathbf{B}$ is $n - 1$. What do we then know about the order of $\mathbf{B}$? (I.e., about the numbers m and $n > m$?)

(b) In part (b), suppose instead that the equation system $\mathbf{Bx} = \mathbf{0}$ has solution with (precisely!) *two* degrees of freedom.

- What does that tell you about the rank r of $\mathbf{B}$?
- Let $\mathbf{B}$ have rank r as in the previous bullet item, and suppose that $r = m$. Do we then know enough to conclude that whatever m-vector $\mathbf{b}$, the equation system $\mathbf{Bx} = \mathbf{b}$ also has solution with (precisely) two degrees of freedom?

(c) Suppose the problem $\max\ \mathbf{x}'\mathbf{S}\mathbf{x}$ subject to $\mathbf{Bx} = \mathbf{b}$ has a solution, call it $\mathbf{x}^*$. Decide *true or false* for each of the two statements:

I. «$\mathbf{S}$ cannot be positive definite.» (You can put $\mathbf{b} = \mathbf{0}$ if you think that simplifies.)

II. «$\mathbf{S} + \mathbf{S}'$ must be negative semidefinite.»

Problem 3. Let $T > K > 0$ be given constants. Consider the problem

$$\max_{u(t)\in[-1,0]} \int_K^T \left(ux^2 - Kx^4\right) dt \qquad \text{where} \qquad \dot{x} = u, \qquad x(K) = K \qquad x(T) \geq 0.$$

(a)
- State the conditions from the maximum principle.
- Suppose that (for given constants $T > K > 0$) there is an admissible *constant* control $u(t) \equiv -c$ that satisfies all the conditions from the maximum principle.

  Do we know (in this course) enough to conclude that this control is optimal?

Hint for (b) and (c): $p(t) + \big(x(t)\big)^2$ can be found using $p(T) + \big(x(T)\big)^2$ and $\frac{d}{dt}\left(p + x^2\right)$. You are allowed to put $K = 1$ in part (b) if that simplifies, and still get full score – but you cannot let $K = 1$ in the next part (c).

(b) For what $T > K$ – if any – will the pair $(x, u) \equiv (K, 0)$ satisfy the conditions from the maximum principle? That is, the constant control $u \equiv 0$ that yields constant x.

(c) Take for granted that $u(t) \equiv -1$ is optimal for some problem that has $T = 2K$. Approximately how much does the optimal value change if T is increased by ϵ?

Problem 4. Let F be a production function defined for $K > 0$ and $L > 0$, satisfying $F > 0$, $F'_K > 0$ and $F'_L > 0$. Suppose furthermore that its elasticity of substitution satisfies the form $\sigma_{L,K} = F \cdot \frac{F'_K F'_L}{KL} \Big/ \begin{vmatrix} 0 & F'_K & F'_L \\ F'_K & F''_{KK} & F''_{KL} \\ F'_L & F''_{KL} & F''_{LL} \end{vmatrix}$ and is > 0; you can take for granted that this form implies *constant returns to scale, i.e.:* $F(tK, tL) = tF(K, L) > 0$, all $t > 0,\ K > 0,\ L > 0$.

(a) Why do the above assumptions imply that F is quasiconcave?

(b) Must F be concave? Justify one of the possible answers «must be concave» or «need not be concave» or point out that «there is no way to tell using this course».

Problem 5. Let p, q, z and b_t be constants, all $\in (0, 1)$. Consider the problem

$$J_{t_0}(x) = \max_{u_t\in[0,1]} \left\{ b_T x_T + \sum_{t=t_0}^{T-1} b_t u_t^p x_t \right\} \qquad \text{where } x_{t+1} = (1 - u_t^q)x_t, \quad x_{t_0} = z$$

(a)
- Suppose $p \neq q$. Calculate the optimal u^*_{T-1} and u^*_{T-2}, assuming that they are determined by first-order conditions. (I.e., you can use the FOC without checking that the constants are such that this approach yields maximum.)
- What happens to u^*_{T-1} when $p = q$? A verbal description would suffice.

(b) Show by induction that as long as $x_t > 0$, the following hold true for all $t = t_0, \ldots, T-1$ – regardless of whether the u^*_t are determined by first-order conditions:
- $J_t(x_t) = A_t x_t$, where A_t does not depend on x_t
- the optimal u^*_t (which you are free to assume exists!) does not depend on x_t.