ECON4140 Mathematics 3: postponed exam 2020-08-10

There are 2 pages of problems to be solved.
All printed and written material may be used, as well as both the approved calculators.
Grades given run from A (best) to E for passes, and F for fail.

- You are required to state reasons for all your answers.
- You are permitted to use any information stated in an earlier letter-enumerated item (e.g. "(a)") to solve a later one (e.g. "(c)"), regardless of whether you managed to answer the former. A later item does not necessarily require answers from or information given in a previous one.

Problem 1. Let $a > 0 \geq b$ be constants, and let $\mathbf{A} = \begin{pmatrix} a & b \\ a & b \end{pmatrix}$.

(a) Is $\mathbf{u} = (-b,\ a)'$ an eigenvector for $\mathbf{A}$? (Note that $\mathbf{u} \neq \mathbf{0}$, as $a > 0$.)

(b) Find an eigenvector associated to the eigenvalue $\lambda = a + b$.

(c) Note how λ equals the ratio $\rho = \dfrac{\text{sum of elements}}{\text{number of rows}}$. Decide which statement is true:

- ☐ Every $n \times n$ matrix has an eigenvalue that equals this ratio ρ.
- ☐ Every $n \times n$ matrix has an eigenvalue with real part equal to that ratio ρ, even if a real eigenvalue may fail to exist when n is even.
- ☐ The previous statement is true for $n = 2$ (and $n = 1$!), but false for $n \geq 3$.
- ☐ No such rule for $n > 1$; such an eigenvalue is a coincidence.

Consider from now on the the differential equation system (D) given by

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \mathbf{A} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} t + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \sin(e\pi t) \tag{D}$$

(d) *Without* solving: is it possible from parts (a), (b) and (c) to know whether (D) is globally asymptotically stable? (The answer might depend on a and b.)

(e) Consider again system (D):

- *Deduce* – write out the calculations in detail! – a second-order differential equation for y (note, for «y» and not x);
- *Explain how* to find the general solution of that differential equation when $a > |b|$.

Problem 2. Let $\mathbf{B}$ be a matrix of order $m \times n$, where $m < n$.

(a) In part (a), suppose that the rank of $\mathbf{B}$ is $n-1$. What do we then know about the order of $\mathbf{B}$? (I.e., about the numbers m and $n > m$?)

(b) In part (b), suppose instead that the equation system $\mathbf{Bx} = \mathbf{0}$ has solution with (precisely!) *two* degrees of freedom.

- What does that tell you about the rank r of $\mathbf{B}$?
- Let $\mathbf{B}$ have rank r as in the previous bullet item, and suppose that $r = m$. Do we then know enough to conclude that whatever m-vector $\mathbf{b}$, the equation system $\mathbf{Bx} = \mathbf{b}$ also has solution with (precisely) two degrees of freedom?

Problem 3. Let $T > K > 0$ be given constants. Consider the problem

$$\max_{u(t)\in[-1,0]} \int_K^T \left(ux^2 - Kx^4\right) dt \qquad \text{where} \qquad \dot{x} = u, \qquad x(K) = K \qquad x(T) \geq 0.$$

(a)
- State the conditions from the maximum principle.
- Suppose that (for given constants $T > K > 0$) there is an admissible *constant* control $u(t) \equiv -c$ that satisfies all the conditions from the maximum principle.

  Do we know (in this course) enough to conclude that this control is optimal?

Hint for (b) and (c): $p(t) + \big(x(t)\big)^2$ can be found using $p(T) + \big(x(T)\big)^2$ and $\frac{d}{dt}\left(p + x^2\right)$. You are allowed to put $K = 1$ in part (b) if that simplifies, and still get full score – but you cannot let $K = 1$ in the next part (c).

(b) For what $T > K$ – if any – will the pair $(x, u) \equiv (K, 0)$ satisfy the conditions from the maximum principle? That is, the constant control $u \equiv 0$ that yields constant x.

(c) Take for granted that $u(t) \equiv -1$ is optimal for some problem that has $T = 2K$. Approximately how much does the optimal value change if T is increased by ϵ?

Problem 4. Let F be a production function defined for $K > 0$ and $L > 0$, satisfying $F > 0$, $F'_K > 0$ and $F'_L > 0$. Suppose furthermore that its elasticity of substitution satisfies the form $\sigma_{L,K} = F \cdot \frac{F'_K F'_L}{KL} \Big/ \begin{vmatrix} 0 & F'_K & F'_L \\ F'_K & F''_{KK} & F''_{KL} \\ F'_L & F''_{KL} & F''_{LL} \end{vmatrix}$ and is > 0; you can take for granted that this form implies *constant returns to scale, i.e.:* $F(tK, tL) = tF(K, L) > 0$, all $t > 0,\ K > 0,\ L > 0$.

(a) Why do the above assumptions imply that F is quasiconcave?

(b) Must F be concave? Justify one of the possible answers «must be concave» or «need not be concave» or point out that «there is no way to tell using this course».