University of Oslo / Department of Economics / NCF

ECON4140 Mathematics 3 – on the 2021–05–26 exam

- *Standard disclaimer:*
  - This note is not suited as a complete solution or as a template for an exam paper. It was written as guidance for the grading process – however, with additional notes and remarks for using the document in teaching later.
  - The document reflects what was expected in that particular semester, and which may not be applicable to future semesters.
- *Weighting:* The problem set was written with the intention that a uniform weighting over letter-enumerated items should be a *feasible* choice. The grading committee – and in case of appeals, the appeals committee – can decide otherwise.
  - Problem 1 covers two topics, namely linear algebra and differential equations. The apparently high weight (5/11 if uniform) for Problem 1, was intentional.
- *Default grade intervals:* Default percent score to grade conversion table for this course:

F (fail)	E	D	C	B	A
0 to 39	40 to 44	45 to 54	55 to 74	75 to 90	91 to 100

  The committee (and in case of appeals, the new committee) is free to deviate.
- *Recent "deviations from defaults" for reference:* 2016 through 2018 have had post-grading information available, indicating deviations from weighting and conversion table and other measures made. Brief overview with links to documents provided:
  - 2016: Weighting and conversion table kept at default, but certain elementary errors and misconceptions got a more forgiving treatment.
  - 2017: The committee considered the exam a bit easy, although the threshold for A *«was practiced slightly leniently in order to distinguish out the best»*.
  - 2018: Weighting was tweaked (for the benefit of a very few papers) and the committee *«stretched the "A" threshold a bit downwards»*.
  - 2019: Default grades applied. Some considerations on one letter item.
  - 2020: Was a take-home exam, 2020 considerations likely not relevant here.

Notes The next pages will give remarks to the problem set in order. At the end, a close-to-minimal handwritten solution is attached. There is no requirement to be as to-the-point as that – indeed, that brevity is not even recommended.

Most likely, 1(d) and 4(c) will be the hardest parts – although I expect some not-watertight arguments in 3(b), like forgetting all about the sign of B, trying to invoke an extreme value theorem without going by way of $[\epsilon, 1-\epsilon]$ and appealing to a FOC without taking note of the $(0, Q]$ restriction despite the reminder.

Remarks to problem 1

(a)
- The right answer is the third option with the correct justification.
- A wrong but not at all bad answer would be to calculate $\mathbf{Au} = e^q\mathbf{u}$ and conclude «always», forgetting about the $q = 0$ exception.
- A truly bad answer is to conclude the third option based on the wrong idea that $q = 0$ makes for an eigenvector as $q = 0$ makes $\mathbf{Au} = \mu\mathbf{u}$ hold.

(b) Intentional language: calculate «an» associated eigenvalue. No need to talk about scalings.

(c) Trace is the shortest question. Also it can be calculated using product = determinant, but that is actually not within what asked in neither first nor second bullet item; exercise judgement.

(d) Of course it is only due to the specific form of $\mathbf{A}$ that an eigenvalue for the top-left 2×2 block is $(v_1, v_2)'$, but I would expect (and accept) papers to jump at that – especially given the hint.

The question is deliberately formulated so that it does not require to know that $\lambda = -2$ is the *only* negative eigenvalue; it would work if the origin were a sink and not a saddle.

Remarks to problem 2 Problems like these are given from time to time; often, if a paper makes the mistake of stating same solution (or same constant solution), the differential equation solution is stated. This time, the difference equation problem appears first. Maybe that induces other answers.

Although using real and imaginary parts is encouraged, we are asking for real solutions. That is, for full score one should not write answers as complex exponentials. Using the shift form $C\cos(\omega + \theta t)$ is just fine.

As for part (c), the major point is that both cos and sin are needed; it was intentional not to ask to calculate constants. It was also intentional to keep the «20» for them to demonstrate that a sum on the RHS translates into a sum of functions, but the committee should exercise best judgement to omissions of that constant.

Remarks to problem 3 Two lectures and the last seminar set elaborated on problems where the optimization reduces to a $\sqrt{x} \cdot \max_u h(u)$ form where the $h(u^*)$ does not depend on x. If one can handle basic log rules, the x-dependence splitting off additively should arguably be easier. Harder though is that the log function does not admit the extreme value theorem, hence the hint of sketching a graph.

Also the last lecture covered the trick of augmenting the induction hypothesis as you go.

Remarks to problem 4 It is not expected to even mention the «p_0» constant – the constraint qualification thing. I often consider whether even the problem text should mention it, at the risk of creating confusion. This time I chose not to say anything; my hunch was that omitting it saves more time in total. If a very few papers would want to incorporate it into the arguments, it does not take that much extra.[1]

(a) The maximality condition is not the same as the first-order condition! It is expected that they write out what H is rather than just say that u^* «maximizes H» without any specification, but it is just fine to write e.g. $H(t,x,u,p) = \underbrace{u \cdot [e^{t-x} - p]}_{u^* \text{ maximizes this over } u \geq 0} + \sin(tx) + p \cdot (x-2)$.

The transversality condition on p should not be omitted.

The differential equation for x *can* be omitted from the answer, or it *can* be included – at their choice. It needs to hold, of course, but that is an issue for when we start *applying* the conditions, i.e. part (b).

(b) This question intends to test whether they can apply the conditions (including the differential equation for x); again they cannot use the first-order condition $p = e^{t-x}$.

(c) And again they cannot use the first-order condition $p = e^{t-x}$ without noting that it could fail – but if fails, then $u^* = 0$ which eliminates the term.

The sensitivity in question was raised in the review lecture – by coincidence, by a student – and is still arguably one of the tougher questions in the exam set, in particular when it is sensitivity wrt. initial time that comes with a sign that is easy to forget and hard to debug, given that you are supposed to get zero.

[1] Part (a) works with some more glyphs; in (b), $u \equiv 0$ still yields $x(3) > 1$ yields $p(3) = 0$, so $p_0 = 0$ is also impossible under the $u \equiv 0$ assumption; in (c), the point is still that either u or its first-order coefficient must vanish, and $p_0 \sin 0$ is zero anyway.

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a "minimal" (not recommended) solution

1a): Not "always" as $q=0$ yields $\vec{0}$.

$\bar{A}\vec{u} = \begin{pmatrix} 0+0+0 \\ 0+0+0 \\ 0+0+e^q q \end{pmatrix} = e^q \vec{u}$, eigenvector for $q \neq 0$

(Third option.)

1b): $\bar{A} - (-2)\bar{I} = \begin{pmatrix} 4 & 1 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 2+e^q \end{pmatrix}$ so $v_3 = 0$ and

$(4 \ 1)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$. $(a \ b)\begin{pmatrix} b \\ -a \end{pmatrix} = 0$; $\vec{v} = \begin{pmatrix} 1 \\ -4 \\ 0 \end{pmatrix}$

1c): $e^q + \lambda + \gamma = \operatorname{tr} \bar{A} = 2 - 1 + e^q$, $\gamma = 1 - (-2) = 3$.

1d): Jacobian $= \begin{pmatrix} 2 & 1 \\ 4\cos x & -1 \end{pmatrix}$ becomes $\begin{pmatrix} 2 & 1 \\ 4 & -1 \end{pmatrix}$ at $\vec{0}$, top-left 2x2 block of $\bar{A}$.

from (b)

We see that $\begin{pmatrix} 2 & 1 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ with $\lambda = -2 < 0$

Negative λ. So $\lim \frac{y(t)}{x(t)} = \frac{v_2}{v_1} = \frac{-4}{1} = -4$.

2: Will need r st $r^2 - 2r + \frac{5}{4} = 0$

$r = 1 \pm \frac{1}{2}\sqrt{4 - 4 \cdot \frac{5}{4}} = 1 \pm \frac{1}{2}\sqrt{-1} = 1 \pm \frac{i}{2}$

2a) $u^*_c = Q$ constant s.t. $\underbrace{Q - 2Q + \frac{5}{4}Q}_{Q/4} = 20$ $\quad Q = 80$

Need $\sqrt{b} = \sqrt{\frac{5}{4}} = \frac{1}{2}\sqrt{5}$,

$-\frac{a}{2\sqrt{b}} = \frac{2}{\sqrt{5}} = \frac{2}{5}\sqrt{5} = \cos\theta$. With this θ,

solution $= 80 + (\frac{1}{2}\sqrt{5})^t (C\cos(\theta t) + D\sin(\theta t))$

2b) Constant solution u^* st $\frac{5u^*}{4} = 80$, $u^* = 16$

Real pt $= 1$, im pt $= \frac{1}{2}$. Solution:

$16 + e^t (C\cos\frac{t}{2} + D\sin\frac{t}{2})$

2c) Try $16 + K\cos\frac{t}{\pi} + L\sin\frac{t}{\pi}$, fit K, L.

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4 $H(t, x, u, p) = \sin(tx) + ue^{t-x} + p \cdot (x - 2 - u)$

(a) Cond's: u^* maximizes H, i.e.

- u^* maximizes $u \cdot (e^{t-x} - p)$ over $u \geq 0$
- $\dot{p} = -t\cos(tx) + ue^{t-x} - p$ with
- $p(3) \geq 0$ ($= 0$ if $x^*(3) > 2$)

(b) $u \equiv 0$ yields $\dot{x} = x - x(0)$ yields x constant > 2 so $p(3) = 0$. But $u = 0$ does not maximize $u \cdot (e^{3-2} - 0)$ over $u \geq 0$.

(c) $r'(s) = -H|_{t=s} = -[\sin(sx) + u[e^{s-2} - p] + p \cdot 0]$

One of these must be 0; if $u \neq 0$, $p = e^{t-x}$

$s = 0$: $-[\underbrace{\sin 0}_{0} + 0 + 0]$

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"Minimal" solution [not recommended!]

3b first. OK for $\tau=0$, $A_0=0$, $B_0=\frac{1}{Q}>0$.

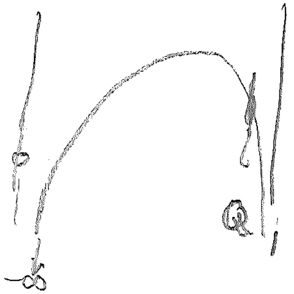
Suppose for induction: true at τ, with $B_\tau>0$.

Then at $\tau+1$:

$$J_{T-(\tau+1)}(x)=\max_{u\in(0,Q]}\left\{\ln(ux)+A_\tau+B_\tau\ln[(1-u^2)x]\right\}$$

$$=\underbrace{(1+B_\tau)}_{B_{\tau+1}>0}\ln x+\underbrace{A_\tau+\max_{u\in(0,Q]}\left\{\ln u+B_\tau\ln(1-u^2)\right\}}_{A_{\tau+1}}$$

Need $A_{\tau+1}$ to exist. Since $B_\tau>0$ [extra assumption], $\ln u+B_\tau\ln(1+u)+B_\tau\ln(1-u)$ concave and



has a max.

For (a): When $Q=1$, it must be stationary!

So (b) **OK** with $B_{\tau+1}=1+B_\tau>0$, $A_{\tau+1}=A_\tau+\max_{u\in(0,Q]}\left\{\ln u+B_\tau\ln(1-u^2)\right\}$

3a J_{T-1}: $\tau=1$:

$$J_{T-1}=(1+1)\ln x+\max_{u\in(0,1]}\ln u+\ln(1-u^2)$$

with $B=1/Q=1$

FOC $\frac{1}{u}=\frac{2u}{1-u^2}$, $1-u^2=2u^2$

$3u^2=1$, $u^*_{T-1}=\frac{1}{3}\sqrt{3}$

$$J_{T-1}(x)=2\ln x+\ln\sqrt{\tfrac{1}{3}}+\ln(1-\tfrac{1}{3})$$

$$=2\ln x+\ln 2-\tfrac{3}{2}\ln 3$$

$$J_{T-2}(x)=3\ln x+\ln 2-\tfrac{3}{2}\ln 3+\max_{u\in(0,1]}\left\{\ln u+2\ln(1-u^2)\right\}$$

FOC: $\frac{1}{u}=\frac{4u}{1-u^2}$ $\quad 1-u^2=4u^2$

$5u^2=1$, so

$$u^*_{T-2}=\frac{1}{5}\sqrt{5}$$