ECON4140 Mathematics 3: exam 2022-05-25, 09:00–12:00

There are 2 pages of problems to be solved.

All printed and written material may be used, as well as both the approved calculators.

Grades given run from A (best) to E for passes, and F for fail.

- You are required to state reasons for all your answers.
- You are permitted to use any information stated in an earlier letter-enumerated item (e.g. "(a)") to solve a later one (e.g. "(c)"), regardless of whether you managed to answer the former. A later item does not necessarily require answers from or information given in a previous one.

Problem 1 Consider the matrix $\mathbf{A} = \begin{pmatrix} 4 & 3 & 9 \\ 1 & 0 & 1 \\ -1 & 2 & 1 \end{pmatrix}$.

It *may or may not* be helpful to know that $\mathbf{A}$ has a rank equal to 3, a determinant equal to 4 and a trace equal to 5.

(a) Show that the vector $\mathbf{u} = (2, 1, -1)'$ is an eigenvector of $\mathbf{A}$.

(b) $\lambda = 2$ is an eigenvalue of $\mathbf{A}$. Find an associated eigenvector $\mathbf{v}$.

(c) Show that every eigenvector of $\mathbf{A}$ is either a scaling of $\mathbf{u}$ or a scaling of $\mathbf{v}$. (I.e. it has *no other* eigenvectors. This part (c) might take longer time than part (a) or (b).)

(d) Decide the definiteness of the quadratic form $Q(\mathbf{x}) = \mathbf{x}'\mathbf{A}\mathbf{x}$.

Problem 2 Consider the differential equation system

$$\dot{x} = -5x + 2y + t, \qquad \dot{y} = -2x - y \tag{S}$$

(a) Deduce a second-order differential equation for x. (Write out the calculations!)

(b) Find the general solution of that differential equation from (a).

(c) Consider the *homogeneous* system corresponding to (S). Sketch a phase diagram and indicate some solution curves.

Problem 3 Consider – but do not try to solve! – the optimal control problem

$$\max_{u(t)\in U} \int_1^{\pi} \left(x\sin(\pi u/6) - \pi x^2\right) dt \qquad \text{subject to} \quad \dot{x} = (u-2)x, \quad x(1) = \pi, \quad x(\pi) \text{ free}$$

where $U = \{0, 1, 2, 3, 4, 5, 6\}$ – i.e. you can choose $u(t)$ among seven *distinct* values. Note that $x(t) > 0$ for any admissible pair (x, u).

(a) State the conditions from the maximum principle.

(b) If an admissible control u^* satisfies the conditions from part (a), what do we then know about $u^*(\pi)$ and (*be precise here!*) about $u^*(t)$ at times t very close to π?

(c) Show that if an admissible pair (x^*, u^*) satisfies the conditions from the maximum principle, then it does indeed solve the problem. (*Hint:* pay attention to the details!)

Problem 4

(a) Let $f_0(t) = 1$ and define inductively the functions $f_{n+1}(t) = -e^{t^2/2}\dfrac{d}{dt}\left[f_n(t)e^{-t^2/2}\right]$. Show that f_n is a polynomial.

(b) Let F be a twice continuously differentiable function of two variables $K > 0$ and $L > 0$, and recall that the elasticity of substitution $\sigma = \sigma(K, L)$ can be written as

$$\sigma(K,L) = \frac{F_K' F_L'}{KL} \cdot \frac{KF_K' + LF_L'}{B(K,L)} \quad \text{where} \quad B = \begin{vmatrix} 0 & F_K' & F_L' \\ F_K' & F_{KK}'' & F_{KL}'' \\ F_L' & F_{KL}'' & F_{LL}'' \end{vmatrix}$$

In certain economic applications you often see functions that satisfy the condition

$$\text{For all } K > 0,\ L > 0 \text{ we have} \quad \sigma > 0,\ F_K' > 0, \text{ and } F_L' > 0 \qquad (*)$$

and the question is to draw the right implication from that condition. That is:

Question: *Precisely one of the following statements is true. Which one?*

- ☐ If (*) holds, we know that F is quasiconvex (but it need not be convex).
- ☐ If (*) holds, we know that F is quasiconcave (but it need not be concave).
- ☐ If (*) holds, we know that F is concave.
- ☐ None of the above hold generally. (That is: Although many production functions satisfy both (*) and quasiconvexity and concavity, none of the latter two conditions follow mathematically from (*).)

(end of problem set)