University of Oslo / Department of Economics / NCF

ECON4140 Mathematics 3 – on the 2022–05–25 exam

- *Standard disclaimer:*
  - This note is not suited as a complete solution or as a template for an exam paper. It was written as guidance for the grading process – however, with additional notes and remarks for using the document in teaching later.
  - The document reflects what was expected in that particular semester, and which may not be applicable to future semesters.
- *Weighting:* The problem set was written with the intention that a uniform weighting over letter-enumerated items should be a *feasible* choice. The grading committee – and in case of appeals, the appeals committee – can decide otherwise.
- *Default grade intervals:* Default percent score to grade conversion table for this course:

F (fail)	E	D	C	B	A
0 to 39	40 to 44	45 to 54	55 to 74	75 to 90	91 to 100

  The committee (and in case of appeals, the new committee) is free to deviate.
- *Recent "deviations from defaults" for reference:* 2016 through 2019 have had post-grading information available, indicating deviations from weighting and conversion table and other measures made. Brief overview with links to documents provided:
  - 2016: Weighting and conversion table kept at default, but certain elementary errors and misconceptions got a more forgiving treatment.
  - 2017: The committee considered the exam a bit easy, although the threshold for A *«was practiced slightly leniently in order to distinguish out the best»*.
  - 2018: Weighting was tweaked (for the benefit of a very few papers) and the committee *«stretched the "A" threshold a bit downwards»*.
  - 2019: Default grades applied. Some considerations on one letter item.

  (2020 was a take-home exam, and so 2020 considerations are likely not as relevant; for 2021 there were still some COVID measures in place, and due to illness, I did not follow up well on 2021.)

In this set, parts 1c, 3b, 3c and 4b (the not-necessarily-concave part) are expected to be the toughest. The committee should exercise judgement if these in total turn out worse than plausibly expected. As for 2(c), phase diagrams often score less than average, which is suspected to be due to time budget considerations; here the system is linear and therefore easier than usual, but still do not be surprised if candidates leave it for last.

The next pages will restate problems as given, followed by (boxed) solutions and remarks. One page per problem, except problem 4.

Problem 1 Consider the matrix $\mathbf{A} = \begin{pmatrix} 4 & 3 & 9 \\ 1 & 0 & 1 \\ -1 & 2 & 1 \end{pmatrix}$.

It *may or may not* be helpful to know that $\mathbf{A}$ has a rank equal to 3, a determinant equal to 4 and a trace equal to 5.

(a) Show that the vector $\mathbf{u} = (2, 1, -1)'$ is an eigenvector of $\mathbf{A}$.

(b) $\lambda = 2$ is an eigenvalue of $\mathbf{A}$. Find an associated eigenvector $\mathbf{v}$.

(c) Show that every eigenvector of $\mathbf{A}$ is either a scaling of $\mathbf{u}$ or a scaling of $\mathbf{v}$. (I.e. it has *no other* eigenvectors. This part (c) might take longer time than part (a) or (b).)

(d) Decide the definiteness of the quadratic form $Q(\mathbf{x}) = \mathbf{x}'\mathbf{A}\mathbf{x}$.

Problem 1: how to solve

(a) $\mathbf{Au} = \begin{pmatrix} 4 \cdot 2 + 3 \cdot 1 - 9 \\ 2 + 0 + (-1) \\ -2 + 2 + (-1) \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \mathbf{u}$, *OK* (with eigenvalue $\mu = 1$).

(b) In $\mathbf{A} - 2\mathbf{I} = \mathbf{0}$, the third equation is the negative of the second, and can be deleted. On the first two, $\begin{pmatrix} 2 & 3 & 9 \\ 1 & -2 & 1 \end{pmatrix} \begin{matrix} \leftarrow^{+} \\ \rfloor_{-2} \end{matrix} \sim \begin{pmatrix} 0 & 7 & 7 \\ 1 & -2 & 1 \end{pmatrix}$ so that $y = -z$ and $x = 2y - z$. Putting e.g. $y = -z = 1$ yields $x = 2 + 1 = 3$ and an eigenvector (corresponding to this eigenvalue) is $\mathbf{v} = (3, 1, -1)'$.

(c) The last eigenvalue is $\texttt{tr}\,(\mathbf{A}) - \lambda - \mu = 5 - 2 - 1 = 2 = \lambda$, so any remaining eigenvector has to be associated to $\lambda = 2$. But from the calculations from (b), there are no others than scalings of $\mathbf{v}$; that equation system has only one degree of freedom.

(d) Symmetrize: $\frac{1}{2}(\mathbf{A} + \mathbf{A}') = \begin{pmatrix} 4 & 2 & 4 \\ 2 & 0 & 3/2 \\ 4 & 3/2 & 1 \end{pmatrix}$ is indefinite because the top–left 2×2 minor is $4 \cdot 0 - 2^2 < 0$.

Note: Even without symmetrizing, $Q(\mathbf{v}) = \mathbf{v}'\mathbf{A}v = 2||\mathbf{v}||^2 > 0$, but that is only enough to disprove negative semidefiniteness – which can be disproven by looking at element a_{11} already. The main diagonal elements do not change upon symmetrization, and $4,\ 0,\ 1$ are alone enough to prove that the form is either positive semidefinite or indefinite. But that is not enough. Also the top–left 2×2 criterion requires symmetrization even when we have the $4 \cdot 0$; if a_{12} were $= -a_{21}$, the minor would have been zero.

Problem 2 Consider the differential equation system

$$\dot{x} = -5x + 2y + t, \qquad \dot{y} = -2x - y \tag{S}$$

(a) Deduce a second-order differential equation for x. (Write out the calculations!)

(b) Find the general solution of that differential equation from (a).

(c) Consider the *homogeneous* system corresponding to (S). Sketch a phase diagram and indicate some solution curves.

Problem 2: How to solve

(a) Differentiating: $\ddot{x} = -5\dot{x} + 2\dot{y} + 1$, which equals $-5\dot{x} + 2(-2x - y) + 1 = -5\dot{x} - 4x + 1 - 2y$. Inserting $2y = \dot{x} + 5x - t$ (or simply adding the first equation) yields $\ddot{x} + \dot{x} = -5\dot{x} - 9x + 1 + t$, or cleaned up: $\ddot{x} + 6\dot{x} + 9x = 1 + t$.

(b) Characteristic polynomial $r^2 + 6r + 9 = (r + 3)^2$ has a single root $r = -3$. Homogeneous equation has general solution $e^{-3t}(At + B)$. We need to add a particular solution. Trying $kt + \ell$ yields $6k + 9(kt + \ell) = 1 + t$ so that $k = 1/9$ and $6/9 + 9\ell = 1$, $\ell = 1/27$. General solution: $x = e^{-3t}(At + B) + \frac{1}{9}t + \frac{1}{27}$.

(c) *(Explained:)* The first test here is to know what the homogeneous system is: delete the t. Then we have nullclines: vertical-only motion where $-5x + 2y = 0$ i.e. the line $y = \frac{5}{2}x$; horizontal-only motion where $y = -2x$. Above the $\dot{x}$-nullcline, y is larger and $\dot{x} > 0$; above the $\dot{y}$-nullcline, $-y$ is smaller and $\dot{y} < 0$. Those two straight lines (linear system!) through the origin partition the plane into four regions. Curves enough to pass through all – together with the nullclines – would suffice. E.g. this plot from Wolfram|Alpha has way more than enough paths, but is lacking the two nullclines, *which are expected for full score.*

Problem 3 Consider – but do not try to solve! – the optimal control problem

$$\max_{u(t)\in U} \int_1^{\pi} \left(x\sin(\pi u/6) - \pi x^2\right) dt \qquad \text{subject to} \quad \dot{x} = (u-2)x, \quad x(1) = \pi, \quad x(\pi) \text{ free}$$

where $U = \{0, 1, 2, 3, 4, 5, 6\}$ – i.e. you can choose $u(t)$ among seven *distinct* values. Note that $x(t) > 0$ for any admissible pair (x, u).

(a) State the conditions from the maximum principle.

(b) If an admissible control u^* satisfies the conditions from part (a), what do we then know about $u^*(\pi)$ and (*be precise here!*) about $u^*(t)$ at times t very close to π?

(c) Show that if an admissible pair (x^*, u^*) satisfies the conditions from the maximum principle, then it does indeed solve the problem. (*Hint:* pay attention to the details!)

Problem 3: Solution and remarks

(a) With $H(t, x, u, p) = (x\sin(\pi u/6) - \pi x^2 + p\cdot(u-2)x$, conditions are:

- u^* maximizes $x\sin(\pi u/6) - \pi x^2 + p\cdot(u-2)x$ over $u \in \{0, 1, 2, 3, 4, 5, 6\}$.
- p satisfies $\dot{p} = -\sin(\pi u^*/6) + 2\pi x - (u^* - 2)p$ with $p(\pi)$ free.

Notes:

- Conditions are for an admissible pair – for the purposes of part (a), the differential equation for x can be included or omitted at the candidate's discretion.
- Here the control region is discrete – no excuse whatsoever to write any first-order condition.
- One can omit from the first condition the terms with no u: u^* maximizes $x\sin(\pi u/6) + pux$. And even, since $x > 0$, u^* will maximize $\sin(\pi u/6) + pu$. Some reasoning like that is likely needed for part (c), but not for part (a).

(b) Because $p(\pi) = 0$, then *at* final time we maximize $\sin(\pi u/6)$, and we are restricted to seven points on the first positive sine half-wave: the sine maximized at $\pi/2$, that is, $u^*(\pi) = 3$.
Near final time we have $p(t) \approx 0$ and the contribution pu will be smaller than what takes to make any other u value attractive: so $u^* = 3$ will still give larger H than $u = 2$ or $= 4$ or anything else.

(c) The maximum value of $\sin(\pi u/6) + pu$ is some function of p only (no x nor t here!) – call it m. Since $x \geq 0$, the maximum value of $x\cdot(\sin(\pi u/6) + pu)$ is then $xm(p)$. The maximized Hamiltonian is then $x\cdot(m(p) - 2) - \pi x^2$ which is concave in x. Arrow applies.

Problem 4

(a) Let $f_0(t) = 1$ and define inductively the functions $f_{n+1}(t) = -e^{t^2/2} \frac{d}{dt}\Big[f_n(t)e^{-t^2/2}\Big]$. Show that f_n is a polynomial.

(b) Let F be a twice continuously differentiable function of two variables $K > 0$ and $L > 0$, and recall that the elasticity of substitution $\sigma = \sigma(K, L)$ can be written as

$$\sigma(K, L) = \frac{F'_K F'_L}{KL} \cdot \frac{KF'_K + LF'_L}{B(K, L)} \quad \text{where} \quad B = \begin{vmatrix} 0 & F'_K & F'_L \\ F'_K & F''_{KK} & F''_{KL} \\ F'_L & F''_{KL} & F''_{LL} \end{vmatrix}$$

In certain economic applications you often see functions that satisfy the condition

$$\text{For all } K > 0,\ L > 0 \text{ we have} \quad \sigma > 0,\ F'_K > 0, \text{ and } F'_L > 0 \qquad (*)$$

and the question is to draw the right implication from that condition. That is:

Question: *Precisely one of the following statements is true. Which one?*

- ☐ If (*) holds, we know that F is quasiconvex (but it need not be convex).
- ☐ If (*) holds, we know that F is quasiconcave (but it need not be concave).
- ☐ If (*) holds, we know that F is concave.
- ☐ None of the above hold generally. (That is: Although many production functions satisfy both (*) and quasiconvexity and concavity, none of the latter two conditions follow mathematically from (*).)

Problem 4: Solution and remarks

(a) f_0 is a polynomial. Suppose for induction that f_N is a polynomial. Then $f_{N+1}(t) = -e^{t^2/2}\big[f'_N(t)e^{-t^2/2} - tf_N(t)e^{-t^2/2}\big] = tf_N(t) - f'_N(t)$. Because f_N is a polynomial, f'_N is also a polynomial and so is tf_N, and so is their difference.

(b) If (*) holds, then B has to be > 0 as well. Sufficient for strict quasiconcavity is that $(-1)^r b_r > 0$ for all leading principal minors of the determinant B of order $r + 1$, and all $r > 1$. For two variables, that amounts to only checking $r = 2$, i.e. B itself, and so we know that F must be (strictly) quasiconcave.
F need not be concave, cf. Cobb–Douglas with large exponents.

Notes:

- Part (b) is a problem often assigned for seminars, though not this semester. It is the only question that touches the post-Easter lectures.

- The first and last alternative are there to check familiarity with the quasiconcavity/quasiconvexity concepts. Given that F is supposed to take two input variables, the level curves the statements suggest should raise suspicions. But the "economics students' gut feeling-level" notion of quasiconcavity is certainly a major part of the question.
- And so for partial score, the larger fraction for part (b) is arguably to know that there is a connection between bordered Hessian and quasiconcavity. For the fact that they need not be concave, the Cobb–Douglas should be well known – and even a handwaving argument like "elasticity of substitution depends only on level curves, not on concavity" should be OK.