ECON4140 Mathematics 3: exam 2023-05-12

Problems / problem pages to be solved: 1 through 5, *not* counting this page.
All printed and written material may be used. A calculator is available in Inspera.
Grades given run from A (best) to E for passes, and F for fail.

- You are required to state reasons for all your answers.
- You are permitted to use any information stated in an earlier letter-enumerated item (e.g. "(a)") to solve a later one (e.g. "(c)"), regardless of whether you managed to answer the former. A later item does not necessarily require answers from or information given in a previous one.
  - In this particular exam set, a matrix of the form of the below defined **A** will appear in several problems.
    If you use knowledge or facts from one Arabic-enumerated problem in another (e.g. 2 in 1 or vice versa) then point out what fact and what part (e.g. «and thus equation (E), by 2(c)»).
- "Suggested" weights: the grading committee is free to deviate.

The following matrix **A** will appear in multiple problems in this exam: For each real number R, define

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ -5/2 & R \end{pmatrix}$$

The boldfaced character **A** will denote this matrix throughout the entire exam set, for various values of the constant R.

(Problems start next page, one page per problem.)

Problem 1 ***Suggested weight:*** *25 %.*

For each real constant R, consider the matrix $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ -5/2 & R \end{pmatrix}$ (as also given on page 0).

(a) Find the rank of $\mathbf{A}$. Specify in particular that single value R_0 for R, which gives a different rank. (*Hint:* $R_0 \neq 7$, cf. part (b).)

(b)
- Show that the vector $\mathbf{u} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ when $R = 7$,

  and
- find the associated eigenvalue μ.

(c)
- How many distinct real eigenvalues does $\mathbf{A}$ have in each of the two cases $R = R_0$ (cf. part (a))and when $R = 7$ (cf. part (b))?
  (The numbers may or may not be the same.)
- Pick an R such that $\mathbf{A}$ has a real eigenvalue $\lambda \neq \mu$, and find an eigenvector $\mathbf{v}$ associated to this λ.
  (Whether you can pick any of the values $R = R_0$ or $R = 7$, depends on the answer to the previous bullet item: if they do not admit such a $\lambda \neq \mu$, you will have to try something else. There is no extra score for making a choice that leads to complicated calculations.)

(d) Decide the definiteness property of the quadratic form $Q(\mathbf{x}) = \mathbf{x}'\mathbf{A}\mathbf{x}$.
(The answer will depend on R.)

Problem 2 ***Suggested weight:*** *25 %.*
Throughout this problem, let R be a real constant. Consider:

the difference equation	$x_{t+2}-(R+1)x_{t+1}+(R+5)x_t=-R^2$	(D)
the differential equation	$\ddot{x}-(R+1)\dot{x}+(R+5)x=-R^2$	(E)
the differential equation system	$\dot{x}=x+2y+R, \qquad \dot{y}=-\frac{5}{2}x+Ry$	(S)

You can take note that the system (S) can be written in matrix form using the same matrix $\mathbf{A}=\begin{pmatrix}1 & 2\\ -5/2 & R\end{pmatrix}$ as in Problem 1 and on page 0, as $\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=\mathbf{A}\begin{pmatrix}x\\ y\end{pmatrix}+\begin{pmatrix}R\\ 0\end{pmatrix}$.

(a) From system (S), deduce (*write out the calculations!*) that x must satisfy the differential equation (E).

(b) Let $R=-1$ in this part:

- Show that the general solution of the differential equation (E) can be written of the form $A\cdot\cos(2(t-2023))+B\cdot\sin(2(t-2023))+Q$, and find Q.
- Explain how to find the general solution if the right-hand side were changed to «$\cos(\pi t)$».

(c) Take for granted that there is a value R_* for R such that the *homogeneous equation corresponding to the difference equation* (D) has general solution of the form

$$2C\cdot(5/2)^t+D\cdot 2^t$$

You are allowed to use the «R_*» symbol without calculating its numerical value:

- When $R=R_*$: Find the particular solution of (D) (the *inhomogeneous* equation!) such that $x_0=0, \quad x_1=2023$.

(d) Let $R=-3$ in this part.

- Is (D) globally asymptotically stable?
- Is (E) globally asymptotically stable?

Problem 3 ***Suggested weight:*** *25 %.*

Consider optimal control problems of the form

$$\max_{u\in(-\infty,\infty)} \int_{2023}^{2024} \left(\frac{5}{4}x^2 - u^2\right) dt, \quad \text{subject to} \quad \dot{x} = x + 2u - 1 \quad \text{where} \quad x(2023) = \bar{x} > 0$$

and where $x(2024)$ is free. (The initial state $\bar{x}$ is a *given* positive number.)

(a)
- State the conditions from the maximum principle.
- If these conditions hold for an admissible candidate (x, u): is there any hope that Arrow's sufficient conditions will apply? (Mangasarian's will *not!*)

(b) Let (x, u) be admissible and satisfy the conditions from part (a), with u being a continuous function of t. *From these conditions,* calculate one of the following quantities – your choice, so pick the easiest one: $u(2023)$ or $u(2024)$ or $x(2024)$.

(c) Your *task* in this part is as follows:

- Show that the conditions in (a) imply the following differential equation: $\ddot{x} + 4x = -1$.

In order to answer it, you are allowed to use without proof the following information – which is also information given in Problem 2 parts (a) and (b); you may or may not need all pieces of information:
The differential equation system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \mathbf{A} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} R \\ 0 \end{pmatrix} \qquad \text{where} \qquad \mathbf{A} = \begin{pmatrix} 1 & 2 \\ -5/2 & R \end{pmatrix}$$

leads to x satisfying the differential equation

$$\ddot{x} - (R + 1)\dot{x} + (R + 5)x = -R^2$$

which in case $R = -1$ has general solution of the form

$$A \cdot \cos(2(t - 2023)) + B \cdot \sin(2(t - 2023)) + Q$$

where Q is a specific number, not a constant of integration.

Problem 4 ***Suggested weight:*** *20 %.*
Let $f(u) = -1$ if $u = 1/2$ and 0 if $u = 1$. Consider the dynamic programming problem

$$J_{t_0}(x) = \max_{u_t \in \{\frac{1}{2}, 1\}} \Big(-x_T + \sum_{t=t_0}^{T-1} f(u) \Big) \qquad \text{subject to} \quad x_{t+1} = u_t x_t, \quad x(t_0) = x \in (0, 4).$$

Note that u_t is either $\frac{1}{2}$ or 1, nothing in between.

To explain the dynamic programming problem, you can at each point in time either leave the state as is (so that $x_{t+1} = x_t$) and pay nothing, or you can pay 1 to reduce the state by fifty percent. At the end you have to pay x_T. You wish to minimize total cost.

The objective of Problem 4 is to show that when $t_0 < T$ and $x < 4$, the value of the program is the following:

$$W(x) = \begin{cases} -x & \text{when} \quad x \in (0, 2], \quad \text{and} \\ -(1 + x/2) & \text{when} \quad x \in (2, 4) \end{cases}$$

You are *not* asked about the (more complicated) expressions when the starting point x is not in the interval $(0, 4)$.

(a) Show that $J_{T-1}(x) = W(x)$ when $x \in (0, 4)$.

(b) Prove by induction that $J_{T-N}(x) = W(x)$ when $x \in (0, 4)$, for all $N = 1, 2, \ldots$.

Problem 5 ***Suggested weight:*** *5 %.*
Is it possible for a convex set S to contain *precisely 2023 (distinct) points*?

(In other formulation: is it possible for a convex S to be of the form $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_{2023}\}$ with all points distinct?)

`(end of problem set)`