ECON4140 Mathematics 3 – on the 2023–05–12 exam

- *Standard disclaimer:*
  - This note is not suited as a complete solution or as a template for an exam paper. It was written as guidance for the grading process – however, with additional notes and remarks for using the document in teaching later.
  - The document reflects what was expected in that particular semester, and which may not be applicable to future semesters.
- *Weighting:* This set had weightings suggested for each of problems 1 through 5.[1] In case of appeals, the appeals committee can decide otherwise.
- *Grade intervals:* Percent score to grade conversion table for this course *defaults to* the first line of the following table. The second line are the Mathematics department's defaults (link in Norwegian), based on recommendations from Norsk matematikkråd

	F (fail)	E	D	C	B	A
ECON4140	0 to 39	40 to 44	45 to 54	55 to 74	75 to 90	91 to 100
Math dept.	0 to 39	40 to 45	46 to 57	58 to 76	77 to 91	92 to 100

  The committee (as well as an appeals committee) can deviate at their discretion.[2]

In this set, expected pitfalls were the following: 1(d) forgetting to symmetrize; 2(b) the «-2023» and possibly also including a $\sin(\pi t)$ term; in problem 2, confusing differential and difference eq. criteria; in 3(a), Arrow *failing* doesn't happen that often; the split definition of W in Problem 4, especially (b); and, Problem 5 would be a hit-or-miss, thus also the low weight suggested.[3]

The next pages will restate problems as given, followed by (boxed) solutions and remarks.

[1] Addendum after grading: The weights were implemented, although some discretion was applied on the letter-enumerated parts; in the end, those considerations did not affect any grade.

[2] Addendum after grading: The committee did find this set to be on the easy side, as a good student would be able to solve the majority of questions very quickly and have more than enough time to look up. One aimed for the stricter scale, at no impact. The exam was attended by a small self-selected group, likely explaining the sizeable proportion of good papers and the skewed grade distribution.
For reference follow some considerations from post-grading information available for 2016 through 2019:
- 2016: Weighting and conversion table kept at default, but certain elementary errors and misconceptions got a more forgiving treatment.
- 2017: The committee considered the exam a bit easy, although the threshold for A *«was practiced slightly leniently in order to distinguish out the best»*.
- 2018: Weighting was tweaked (for the benefit of a very few papers) and the committee *«stretched the "A" threshold a bit downwards»*.
- 2019: Default grades applied. Some considerations on one letter item.

(2020 was a take-home exam, and so 2020 considerations are likely not as relevant; for various reasons, the course responsible did not follow up the 2021 and 2022 grading committees.)

[3] Addendum after grading: also, people forgot to argue that the rank of $\mathbf{A}$ is never 0. Furthermore, there is «always» someone who will not *write out* the Hamiltonian (or Lagrangian in Math2), but the committee awarded full score on 3(a) if the written-out conditions were found in 3(b)/3(c).

Problem 1 ***Suggested weight:*** *25 %.*

For each real constant R, consider the matrix $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ -5/2 & R \end{pmatrix}$ (as also given on page 0).

(a) Find the rank of $\mathbf{A}$. Specify in particular that single value R_0 for R, which gives a different rank. (*Hint:* $R_0 \neq 7$, cf. part (b).)

(b)
- Show that the vector $\mathbf{u} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ when $R = 7$,

  and
- find the associated eigenvalue μ.

(c)
- How many distinct real eigenvalues does $\mathbf{A}$ have in each of the two cases $R = R_0$ (cf. part (a))and when $R = 7$ (cf. part (b))?
  (The numbers may or may not be the same.)
- Pick an R such that $\mathbf{A}$ has a real eigenvalue $\lambda \neq \mu$, and find an eigenvector $\mathbf{v}$ associated to this λ.
  (Whether you can pick any of the values $R = R_0$ or $R = 7$, depends on the answer to the previous bullet item: if they do not admit such a $\lambda \neq \mu$, you will have to try something else. There is no extra score for making a choice that leads to complicated calculations.)

(d) Decide the definiteness property of the quadratic form $Q(\mathbf{x}) = \mathbf{x}'\mathbf{A}\mathbf{x}$.
(The answer will depend on R.)

Problem 1: how to solve

(a) The rank is $= 2$ as long as determinant $\neq 0$, i.e. as long as $R \neq -5 =: R_0$. For $R = R_0 = -5$, the rank is < 2, but also > 0 as the matrix is non-null. We have rank $= 1$ when $R = -5$.

(b) Put $R = 7$ and calculate $\mathbf{A}\mathbf{u} = \begin{pmatrix} 2+2 \\ -5+7 \end{pmatrix} = 2\mathbf{u}$, OK with $\mu = 2$.

(c)
- When $R = R_0$ and the determinant is zero, 0 is an eigenvalue. They sum to trace, so the other eigenvalue $1 + R_0 - 0 = 1 - 5$ is distinct from 0 and we have two distinct ones. When $R = 7$ and $\mu = 2$ is an eigenvalue, the other is $1 + 7 - 2 = 6$, so also here we have two distinct ones.
- Choosing the eigenvalue 0 when $R = R_0$, it suffices to find a nonzero vector $(x, y)'$ such that $(1, 2) \cdot (x, y) = 0$. The vector $(2, -1)'$ does the job.

(d) $\frac{1}{2}(\mathbf{A}+\mathbf{A}') = \begin{pmatrix} 1 & 1/4 \\ 1/4 & R \end{pmatrix}$ has determinant $R - 1/16$. It is indefinite when $R < 1/16$. When $R \geq 1/16$, both the 1×1 principal minors are > 0, and so the form is positive definite when $R > 1/16$ and positive semidefinite when $R = 1/16$.

Problem 2 ***Suggested weight:*** *25 %.*
Throughout this problem, let R be a real constant. Consider:

the difference equation	$x_{t+2} - (R+1)x_{t+1} + (R+5)x_t = -R^2$	(D)
the differential equation	$\ddot{x} - (R+1)\dot{x} + (R+5)x = -R^2$	(E)
the differential equation system	$\dot{x} = x + 2y + R, \qquad \dot{y} = -\frac{5}{2}x + Ry$	(S)

You can take note that the system (S) can be written in matrix form using the same matrix $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ -5/2 & R \end{pmatrix}$ as in Problem 1 and on page 0, as $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} R \\ 0 \end{pmatrix}$.

(a) From system (S), deduce (*write out the calculations!*) that x must satisfy the differential equation (E).

(b) Let $R = -1$ in this part:

- Show that the general solution of the differential equation (E) can be written of the form $A \cdot \cos(2(t-2023)) + B \cdot \sin(2(t-2023)) + Q$, and find Q.
- Explain how to find the general solution if the right-hand side were changed to «$\cos(\pi t)$».

(c) Take for granted that there is a value R_* for R such that the *homogeneous equation corresponding to the difference equation* (D) has general solution of the form

$$2C \cdot (5/2)^t + D \cdot 2^t$$

You are allowed to use the «R_*» symbol without calculating its numerical value:

- When $R = R_*$: Find the particular solution of (D) (the *inhomogeneous* equation!) such that $x_0 = 0, \quad x_1 = 2023$.

(d) Let $R = -3$ in this part.

- Is (D) globally asymptotically stable?
- Is (E) globally asymptotically stable?

Problem 2: how to solve

(a) $\ddot{x} = \dot{x} + 2\dot{y} = \dot{x} + 2(\frac{-5}{2}x + Ry) = \dot{x} - 5x + R(\dot{x} - x - R) = (1+R)\dot{x} - (R+5)x - R^2$, now subtract the $\dot{x}$ and x terms from both sides.

(b)
- With $R = -1$, the equation reads $\ddot{x} + 4x = -1$. Since $\left(\frac{d}{dt}\right)^2 [\sin(2(t-2023))] = -2^2 \sin(2(t-2023))$ and $\left(\frac{d}{dt}\right)^2 [\sin(2(t-2023))] = -2^2 \sin(2(t-2023))$, the functions solve the homogeneous equation and (E) has general solution $A \cdot \cos(2(t-2023)) + B \cdot \sin(2(t-2023)) + u^*$. Trying $u^* = Q$ (constant), we see it works when $4Q = -1$, so we are done when $Q = -1/4$.
- Try instead $u^* = k\cos(\pi t) + \ell \sin(\pi t)$ and fit k and ℓ.

(c) Here too we have a constant solution P, but now it satisfies $P - (R_* + 1)P + (R_* + 5)P = -R_*^2$, i.e. $P = -R_*^2/5$. We fit the constants C and D to equations

$$\begin{aligned} &(t = 0:) && 2C + D - R_*^2/5 = 0 \\ &(t = 1:) && 2C \cdot (5/2) + D \cdot 2 - R_*^2/5 = 2023 \end{aligned}$$

Subtracting two of the first from the second yields $C + R_*^2/5 = 2023$, and that in turn yields $D = R_*^2/5 - 2C = 3R_*^2/5 - 4046$. Insert to get

$$2\left(2023 - \tfrac{1}{5}R_*^2\right) \cdot (5/2)^t + \left(\tfrac{3}{2}R_*^2 - 4046\right) \cdot 2^t + \tfrac{1}{5}R_*^2$$

(d) When $R = -3$, the characteristic polynomial takes the form $m^2 + 2m + 2$. Either of the two following arguments would suffice, one does not need both:
[Concluding from roots:] Roots where $(m+1)^2 = -1$ i.e. $m = -1 \pm i$ which has negative real part but modulus/length $> |-1| = 1$, and so the differential equation is globally asymptotically stable while the difference equation is not.
[Concluding from coefficients:] The coefficients are 2 and 2, both positive, so the differential equation is globally asymptotically stable. For the difference equation, the constant term 2 is > 1, which alone is enough to conclude it is not globally asymptotically stable.

Problem 3 ***Suggested weight:*** *25 %.*

Consider optimal control problems of the form

$$\max_{u\in(-\infty,\infty)} \int_{2023}^{2024} \left(\frac{5}{4}x^2 - u^2\right) dt, \quad \text{subject to} \quad \dot{x} = x + 2u - 1 \quad \text{where} \quad x(2023) = \bar{x} > 0$$

and where $x(2024)$ is free. (The initial state $\bar{x}$ is a *given* positive number.)

(a)
- State the conditions from the maximum principle.
- If these conditions hold for an admissible candidate (x, u): is there any hope that Arrow's sufficient conditions will apply? (Mangasarian's will *not!*)

(b) Let (x, u) be admissible and satisfy the conditions from part (a), with u being a continuous function of t. *From these conditions,* calculate one of the following quantities – your choice, so pick the easiest one: $u(2023)$ or $u(2024)$ or $x(2024)$.

(c) Your *task* in this part is as follows:

- Show that the conditions in (a) imply the following differential equation: $\ddot{x} + 4x = -1$.

In order to answer it, you are allowed to use without proof the following information – which is also information given in Problem 2 parts (a) and (b); you may or may not need all pieces of information:
The differential equation system

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \mathbf{A}\begin{pmatrix}x\\ y\end{pmatrix} + \begin{pmatrix}R\\ 0\end{pmatrix} \qquad \text{where} \qquad \mathbf{A} = \begin{pmatrix}1 & 2\\ -5/2 & R\end{pmatrix}$$

leads to x satisfying the differential equation

$$\ddot{x} - (R+1)\dot{x} + (R+5)x = -R^2$$

which in case $R = -1$ has general solution of the form

$$A \cdot \cos(2(t - 2023)) + B \cdot \sin(2(t - 2023)) + Q$$

where Q is a specific number, not a constant of integration.

Problem 3: how to solve Let $H(x,u,p) = \frac{5}{4}x^2 - u^2 + p\cdot(x+2u-1)$.

(a) Conditions:

An optimal u^* must maximize H, i.e. maximize $-u^2+2pu$, over all real u. (Note, need not be pointed out: because u is unrestricted, and $u \mapsto H$ is concave, this is equivalent to $u^* = p$.)

$\dot{p} = -\frac{5}{2}x - p$ with $p(2024) = 0$

It need not be pointed out in the answer to (a), but surely the differential equation must hold too.

The maximized Hamiltonian $H(x,p) \overset{\text{missing } 5/4}{=} x^2 + px - p + \max_u \left\{pu - u^2\right\} \overset{\text{missing } 5/4}{=} x^2 + px +$ [something constant in x] is strictly convex in x, and so not concave, and Arrow's conditions will not apply.

2024 correction: The $\frac{5}{4}$ before x^2 in H was missing. Does not change conclusion.

(b) Because $p(2024) = 0$, $u^*(2024)$ maximizes $-u^2$, so $u^*(2024) = 0$.

(c) With $u^* = p$, we are led to the differential equations $\dot{x} = x + 2p - 1$ and $\dot{p} = -\frac{5}{2}x - p$, which is system (S) with $R = -1$, except with the letter «p» in place of y. But this system implies (E). Inserting $R = -1$ in (E), we get $\ddot{x} + 4x = -1$.

Problem 5 fits the rest of this page:

Problem 5 ***Suggested weight:*** *5 %.*
Is it possible for a convex set S to contain *precisely 2023 (distinct) points*?

(In other formulation: is it possible for a convex S to be of the form $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_{2023}\}$ with all points distinct?)

Problem 5: how to solve Sufficient for full score would be e.g. *«Impossible – such a set must contain an entire line segment, i.e. infinitely many distinct points.»*

The «2023» was deliberately used to frame it as «a lot but finitely many» and it was not intended to require considerations about singletons (or the empty set). At least given the precision level this semester, it is fair to say that one could rightfully think that the above sentence hits spot-on what the problem intended to test.

Problem 4 ***Suggested weight:*** *20 %.*
Let $f(u) = -1$ if $u = 1/2$ and 0 if $u = 1$. Consider the dynamic programming problem

$$J_{t_0}(x) = \max_{u_t \in \{\frac{1}{2},1\}} \Big(- x_T + \sum_{t=t_0}^{T-1} f(u) \Big) \qquad \text{subject to} \quad x_{t+1} = u_t x_t, \quad x(t_0) = x \in (0,4).$$

Note that u_t is either $\frac{1}{2}$ or 1, nothing in between.

To explain the dynamic programming problem, you can at each point in time either leave the state as is (so that $x_{t+1} = x_t$) and pay nothing, or you can pay 1 to reduce the state by fifty percent. At the end you have to pay x_T. You wish to minimize total cost.

The objective of Problem 4 is to show that when $t_0 < T$ and $x < 4$, the value of the program is the following:

$$W(x) = \begin{cases} -x & \text{when} \quad x \in (0,2], \quad \text{and} \\ -(1 + x/2) & \text{when} \quad x \in (2,4) \end{cases}$$

You are *not* asked about the (more complicated) expressions when the starting point x is not in the interval $(0,4)$.

(a) Show that $J_{T-1}(x) = W(x)$ when $x \in (0,4)$.

(b) Prove by induction that $J_{T-N}(x) = W(x)$ when $x \in (0,4)$, for all $N = 1, 2, \ldots$.

Problem 4: how to solve Here part (a) is the base case of the induction. At $T-1$ we are to solve $J_{T-1}(x) = \max\{-f(u) - ux\}$ where we maximize over $\{1/2, 1\}$. $u = 1/2$ would yield $-1 - x/2$ and $u = 1$ would yield $-x$, and (graphically or otherwise), $-x$ is the greatest when $-x \geq -1 - x/2$ i.e. when $x/2 < 1$ i.e. when $x < 2$. Within the range we are asked, this verifies the base case and answers (a).

Suppose true at N. Then at $N+1$ – i.e. at time $T - (N+1)$ – we are to find $\max_u\{-f(u) + W(ux)\}$, i.e. find the largest among $-1 + W(x/2)$ and $W(x)$. Splitting:

- When $x \leq 2$, $W(x/2) = -x/2$ and so $-1 + W(x/2) = -1 - x/2$. Compare this to $W(x) = -x$, which is the greatest when $-x \geq -1 - x/2$ which is true for all $x \in (0,2]$. So $V(x) = W(x)$ on that interval.
- When $x \in (2,4)$, we have $W(x/2) = -x/2$ still (because $x/2 < 2$), and so $-1 + W(x/2) = -1 - x/2$ which now coincides with $W(x)$. So we are to choose between $W(x)$ and $W(x)$, and obviously the max among those is $W(x)$.

Anyway it equals $W(x)$, so we are done.

(Problem 5: see previous page.)