ECON4140 Mathematics 3: exam 2024-05-22

Problems / problem pages to be solved: 1 through 5, *including* this page.
All printed and written material may be used. A calculator is available in Inspera.
Grades given run from A (best) to E for passes, and F for fail.

- You are required to state reasons for all your answers.
- You are permitted to use any information stated in an earlier letter-enumerated item (e.g. "(a)") to solve a later one (e.g. "(c)"), regardless of whether you managed to answer the former. A later item does not necessarily require answers from or information given in a previous one.

Suggested weighting (the grading committee can deviate at their discretion):
1/12 for Problem 5 and for each letter-enumerated item, *except:* Problem 1 counting 2/12 in total.

Problem 1 Consider the matrix $\mathbf{M} = \begin{pmatrix} Q & R \\ S & -Q \end{pmatrix}$, where Q, R, and S are real constants, *not all zero.*

(a) Complete the statement (with justification!):

The rank of $\mathbf{M}$ *is* `<fill in number>` *except if* `<fill in condition>`

(b)
- If $R \cdot S = 0$, what is/are the eigenvalue(s) of $\mathbf{M}$?
- Show that if $Q = 3$ or -3, and $R \cdot S = 16$, then -5 is an eigenvalue.

(c) Are there values of the constants such that $\mathbf{M}$ has no real eigenvalues?

Problem 2 * Throughout this problem, assume $\mathbf{M} = \begin{pmatrix} Q & R \\ S & -Q \end{pmatrix}$ to be invertible.

Consider first the linear differential equation system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \mathbf{M} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(a)
- Deduce a second-order differential equation for $x(t)$ for all Q, R, S. (Carry out the calculation in detail!)
- Find its general solution when $Q = 3$, $R = 16$, $S = 1$. (*Hint* in footnote.)

Let $\phi(x)$ be a twice continuously differentiable function, and let E be a constant. Consider differential equation systems of the following form, valid where $y > 0$:

$$\dot{x} = \phi(x) - y^{-2}, \qquad \dot{y} = E - y \cdot \phi'(x) \tag{*}$$

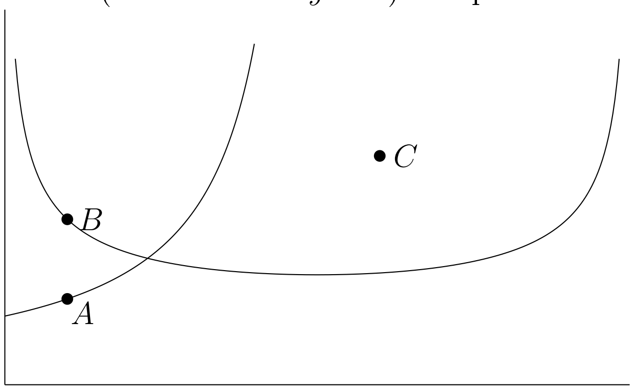
(b) Suppose system (*) has an equilibrium point (i.e. a stationary state) $(\tilde{x}, \tilde{y})$. Show that if $\phi''(\tilde{x}) < 0$ then $(\tilde{x}, \tilde{y})$ must be a saddle point. (Do not try to *find* $(\tilde{x}, \tilde{y})$.)

From now on, let $\phi(x) = (5 - x) \cdot x$ and $E = \frac{3}{2}$ so that (*) becomes

$$\dot{x} = (5 - x) \cdot x - y^{-2}, \qquad \dot{y} = \frac{3}{2} - y \cdot (5 - 2x) \tag{**}$$

(c) Find the limit (as $t \to +\infty$) of the slope of the non-constant particular solutions which converge to the saddle point $(\tilde{x}, \tilde{y}) = (1, \frac{1}{2})$.

The following plot indicates the nullclines of system (**) in the part of the phase diagram where $0 < x < 5$, $0 < y < 1.5$. Three points A (lowest), B and C are given with bullet marks; one *on* each nullcline (both near the y axis) and point C near $(3, 1)$:



(d) Determine and describe the direction of motion at each of these three points A, B and C. (Justify the appropriate signs! You can also indicate with arrows on a copied sketch, but you are *not* asked to draw solution curves.)

*Same matrix $\mathbf{M}$ as in Problem 1, but with the additional condition on the constants that $\mathbf{M}^{-1}$ exists.
- Part (a) might use *the information given in* Problem 1 part (b).
- Part (b) may be harder than part (c); it is possible to solve (c) with or without part (b).

Problem 3 † Let x_0 and T be positive constants. Consider the optimal control problem

$$\max_{u(t)\in[0,14]} \int_0^T \left(u(t) - \frac{3}{2}x(t)\right) dt \quad \text{when} \quad \dot{x} = (5-x)x - \frac{u^2}{4}, \quad x(0) = x_0, \quad x(T) \geq 0.$$

In the following, suppose that (x, u) satisfies the conditions from the maximum principle, with adjoint («costate») process $p = p(t)$.

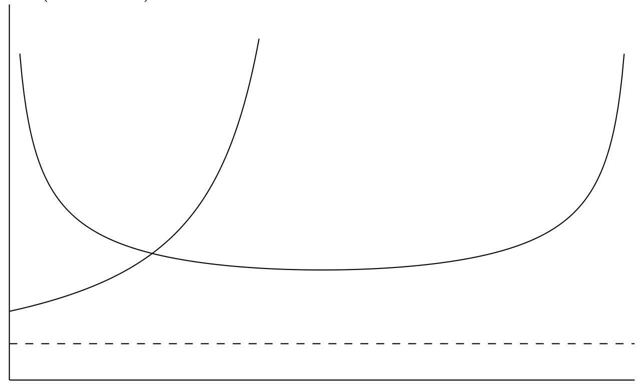
(a)
- State the conditions from the maximum principle.
- Could $u(t)$ be zero? (Warning: $p(t)$ *could* attain negative values.)

(b) Deduce the following differential equation system:

$$\dot{x} = (5-x)x - \left(\frac{1}{\max\{p,\, 1/7\}}\right)^2 \qquad \text{i.e.} = \begin{cases} (5-x)x - 1/p^2 & \text{if } p \geq 1/7 \\ (5-x)x - 49 & \text{if } p < 1/7 \end{cases}$$

$$\dot{p} = \frac{3}{2} - (5-2x)p$$

(c) Take for granted that when $p \geq 1/7$, the system from (b) is the same as system (**) from Problem 2 part (c) with $y = p$, and that the line $p = 1/7$ lies below the nullclines like this (dashed):



- The conditions from the maximum principle restrict the point $(x(T), p(T))$ (that is, at final time T – the «endpoint» of the system!) to a «small» set in this diagram. Which set? I.e., *where could* $(x(T), p(T))$ *possibly be?*
- If furthermore $0 < u(t) < 14$ for all t large enough, $(x(T), p(T))$ is restricted further (i.e. we know more). *Where could* $(x(T), p(T))$ *possibly be in this case?*

Both questions ask for the whole «possible set» as follows from the conditions – not merely an example.

†Though it is surely possible to solve Problem 3 without Problem 2, it cannot be ruled out that some insight from Problem 2 could be useful.

Problem 4 Let $a > 0$ and $p \in (0, 1)$ be constants. Consider the dynamic programming problem

$$J_{t_0}(x) = \max_{u_t \in [0,1]} \Big\{ x_T + a^{1-p} \cdot x_T^p + \sum_{t=t_0}^{T-1} \big(u_t x_t + (u_t x_t)^p\big) \Big\}$$
$$\text{where } x_{t+1} = x_t - u_t x_t, \quad x_{t_0} = x \ (> 0)$$

‡It is possible (and could maybe be quicker) to show (b) first – possibly even with the hint for the more general «bonus» case – and then deduce the answer to (a) from there.

(a) Find $J_{T-2}(x)$ if $a = p = 2/3$.

(b) In this part, allow for more general constants: any $p \in (0, 1)$ and any $a > 0$.

- Show that at time $t = T - k$, we have the form

$$J_{T-k}(x) = x + A_k^{1-p} \cdot x^p \quad \text{where } A_k \text{ does not depend on } x,$$

- Deduce a difference equation for A_k, starting at $A_0 = a$.
(Carry out the maximization to needed to eliminate the «u» letter. Answer will look fairly simple in the end.)

«*Hint and possible bonus*§» How far can you get in the first bullet item if you allow for the following generalizations? (I): The number p might either be $\in (0, 1)$ or > 1, and (II): In place of $u_t \in [0, 1]$, assume $u_t \in U$ where U is closed and bounded and contained in $[0, 1]$

‡The assumptions ensure that $x_t \geq 0$ (and $x_{t-1} \geq x_t$) for all $t = t_0 + 1, \ldots, T$, so $(u_t x_t)^p$ is well-defined. a^{1-p} means a to the $(1 - p)$th power and is just another constant. The form with the $(1 - p)$th power, also occurring in A_k^{1-p} in part (b), might hopefully make some calculations more convenient.

§For (b):
- As «hint»: Depending on your workflow, this could speed up some of your calculations.
- «possible bonus» means that everything but this correct, will be sufficient for 100 percent score – but that the graders *may* at their discretion, take an insightful (brief!) observation here into account on papers that would otherwise be «narrowly missing a grade».

Problem 5 Let f be a function of a real variable, and *continuous* on the entire $\mathbf{R}$. Show that if f is one-to-one, i.e. it has an inverse function f^{-1}, then f^{-1} is quasiconcave.

Hints (if you need them):

- You *will* need a particular property that follows from continuity.[¶] You are allowed to assume continuous differentiability if that makes it simpler, but that property is not essential.
- Recall that an inverse function means $f^{-1}(f(x)) = x$ (all $x \in \mathbf{R}$) and $f(f^{-1}(y)) = y$ (all y in the range of f) – we are not talking about one divided by anything.

`(end of problem set)`

[¶] Indeed if you drop the assumption that f is continuous on its (convex!) domain, the claim is false! Counterexamples without that assumption: $g(t) = 1/t$ defined except at 0: it is its own inverse, but not a quasiconcave function. h defined as $h(t) = g(t)$ and $h(0) = 0$ (thus defined on $\mathbf{R}$) is also its own inverse, but not continuous at 0 and not quasiconcave.