ECON4140 Mathematics 3 – on the 2024–05–22 exam

- *Standard disclaimer:*
  - This note is not suited as a complete solution or as a template for an exam paper. It was written as guidance for the grading process – however, with additional notes and remarks for using the document in teaching later.
  - The document reflects what was expected in that particular semester, and which may not be applicable to future semesters.
- *Weighting:* This set had weightings suggested: 1/12 for Problem 5 and for each letter-enumerated item, *except:* Problem 1 counting 2/12. ‖ In case of appeals, the appeals committee can decide otherwise.
- *Grade intervals:* Percent score to grade conversion table for this course *defaults to* the first line of the following table. The second line are the Mathematics department's defaults (link in Norwegian), based on recommendations from Norsk matematikkråd

	F (fail)	E	D	C	B	A
ECON4140	0 to 39	40 to 44	45 to 54	55 to 74	75 to 90	91 to 100
Math dept.	0 to 39	40 to 45	46 to 57	58 to 76	77 to 91	92 to 100

The committee (as well as an appeals committee) can deviate at their discretion.**

On the next pages, the problems are restated with general remarks (elaborating in part on what seminar problems the questions were related to and *intended* to be relatable to), then how to solve; this time, post-grading remarks were also included at the end of each problem.

‖ Addendum after grading: After reviewing the papers, it might have been, with problems 2(d), 3(c) and 5, that there was too much weight on problems where one had to think a few rounds extra to get how the answer should look. See next footnote.

** Addendum after grading: *In case of a grading appeal, the appeals committee is advised to retrieve sample papers with grades for comparison.*

The thresholds were in practice reduced slightly: Looking back and forth on alternative weightings and alternative thresholds, the committee found it most fair to «generously round up» a couple of papers that were «bar a few details on the same level» as their peers. Given the previous footnote, one should also add that there were «enough easy» questions – maybe more than enough questions in total – but all in all the set turned out harder than expected. The problem set *did* distinguish the better papers from the lesser.

For reference follow some considerations from post-grading information available for 2016 through 2019:

- 2016: Weighting and conversion table kept at default, but certain elementary errors and misconceptions got a more forgiving treatment.
- 2017: The committee considered the exam a bit easy, although the threshold for A *«was practiced slightly leniently in order to distinguish out the best»*.
- 2018: Weighting was tweaked (for the benefit of a very few papers) and the committee *«stretched the "A" threshold a bit downwards»*.
- 2019: Default grades applied. Some considerations on one letter item.

(2020 was a take-home exam, and so 2020 considerations are likely not as relevant; for various reasons, the course responsible did not follow up the 2021 and 2022 grading committees. 2023 was a slightly easier exam, where the stricter scale was attempted at no impact on grades.)

Problem 1 Consider the matrix $\mathbf{M} = \begin{pmatrix} Q & R \\ S & -Q \end{pmatrix}$, where Q, R, and S are real constants, *not all zero.*

(a) Complete the statement (with justification!):

The rank of $\mathbf{M}$ *is* `<fill in number>` *except if* `<fill in condition>`

(b)
- If $R \cdot S = 0$, what is/are the eigenvalue(s) of $\mathbf{M}$?
- Show that if $Q = 3$ or -3, and $R \cdot S = 16$, then -5 is an eigenvalue.

(c) Are there values of the constants such that $\mathbf{M}$ has no real eigenvalues?

Problem 1: Remarks

- The condition that $\mathbf{M} \neq 0$ was mainly to get rid of the exceptional case of rank zero in (a); on a similar question in the 2023 exam, that was too often forgotten. Here part (a) is formulated so that they do not need to address that issue.
- In this year's set, the eigenvalues questions are only for 2×2. One could take note that one needs to find an eigen*vector* in problem 2.
- It might be convenient to already at the start of Problem 1, point out that the determinant is $-(Q^2 + RS)$ and the trace $Q - Q = 0$, and so the eigenvalues will be $\pm\sqrt{Q^2 + RS}$. The «how to solve» below will not do that – instead it will «work as if one reads the problems as they appear and invoke the properties as needed». The answer to (b) second bullet item uses that the eigenvalues (complex, with multiplicity) sum to trace and multiply to determinant, but one could use the formula there as well.

Problem 1: how to solve

(a) *The rank of* $\mathbf{M}$ *is* 2 *except if* $Q^2 + RS = 0$, because nonzero determinant is equivalent full rank (for a square matrix).

(b)
- If $R \cdot S = 0$, the matrix is triangular and the eigenvalues are the main diagonal elements Q and $-Q$ (which may coincide if $Q = 0$).
- Trace is 0, so eigenvalues will be $-\lambda$ and λ (if they exist). With those parameters, $|\mathbf{M}| = -9 - 16 = -25$ which is the product $-\lambda^2$. So $\lambda = \pm 5$.

(c) Eigenvalues: $\frac{1}{2}\operatorname{tr}\mathbf{M} \pm \sqrt{(\frac{1}{2}\operatorname{tr}\mathbf{M})^2 - \det\mathbf{M}}$ which equals $\sqrt{Q^2 + RS}$.
Affirmative: no real eigenvalues if $Q^2 + RS > 0$.

Post-grading remarks: Predicably, Problem 1 was the easiest of the five. Average score within the «A». However, elementary mistakes did occur, like missing one of the two solutions of $\lambda^2 = Q^2$. And, future students, please do not make claims like [redacted] (highlight with mouse to view.)

Problem 2 * Throughout this problem, assume $\mathbf{M} = \begin{pmatrix} Q & R \\ S & -Q \end{pmatrix}$ to be invertible.

Consider first the linear differential equation system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \mathbf{M} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(a)
- Deduce a second-order differential equation for $x(t)$ for all Q, R, S. (Carry out the calculation in detail!)
- Find its general solution when $Q = 3$, $R = 16$, $S = 1$. (*Hint* in footnote.)

Let $\phi(x)$ be a twice continuously differentiable function, and let E be a constant. Consider differential equation systems of the following form, valid where $y > 0$:

(*) $$\dot{x} = \phi(x) - y^{-2}, \qquad \dot{y} = E - y \cdot \phi'(x)$$

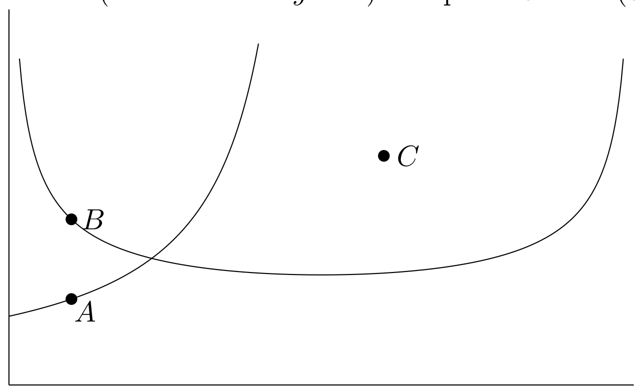
(b) Suppose system (*) has an equilibrium point (i.e. a stationary state) $(\tilde{x}, \tilde{y})$. Show that if $\phi''(\tilde{x}) < 0$ then $(\tilde{x}, \tilde{y})$ must be a saddle point. (Do not try to *find* $(\tilde{x}, \tilde{y})$.)

From now on, let $\phi(x) = (5 - x) \cdot x$ and $E = \frac{3}{2}$ so that (*) becomes

(**) $$\dot{x} = (5 - x) \cdot x - y^{-2}, \qquad \dot{y} = \frac{3}{2} - y \cdot (5 - 2x)$$

(c) Find the limit (as $t \to +\infty$) of the slope of the non-constant particular solutions which converge to the saddle point $(\tilde{x}, \tilde{y}) = (1, \frac{1}{2})$.

The following plot indicates the nullclines of system (**) in the part of the phase diagram where $0 < x < 5$, $0 < y < 1.5$. Three points A (lowest), B and C are given with bullet marks; one *on* each nullcline (both near the y axis) and point C near $(3, 1)$:



(d) Determine and describe the direction of motion at each of these three points A, B and C. (Justify the appropriate signs! You can also indicate with arrows on a copied sketch, but you are *not* asked to draw solution curves.)

*Same matrix $\mathbf{M}$ as in Problem 1, but with the additional condition on the constants that $\mathbf{M}^{-1}$ exists.
- Part (a) might use *the information given in* Problem 1 part (b).
- Part (b) may be harder than part (c); it is possible to solve (c) with or without part (b).

Problem 2: Remarks

- For part (a) it might be useful to know from 1(b) that -5 is an eigenvalue, but it also requires that 5 is an eigenvalue. It should be easy to solve without Problem 1.
- The system form (*) is not that much unlike what shows up in some optimal control problems including the Ramsey–Cass–Koopmans model and a not-too-different resource extraction problem, both covered in class. Of course the form without numbers makes it nontrivial. The key to (b) is that negative determinant at the stationary state implies it is a saddle point.
- One reason why the linear algebra problem 1 is so short, is that one needs to find an eigenvector in Problem 2 (part (c)).
  Problem (c) is not at all uncommon (most recent ordinary exam that assigned it: 2021), and while it not «hard» as in the sense of «requiring any creativity», the theory needed is *several steps up from Math 2,* and it is expected to separate better papers from worse.
- The committee(s) should be aware that part (d) is of a form that has never been given before, intended to make it quicker to solve yet still extract information about the candidates' knowledge. Several exams have asked to sketch a phase diagram (especially the earlier though), and that does take exam time. Thus, sets 2017 and 2020 gave plots with hints about what nullclines could look like (and, in Inspera one problem takes up a page anyway, so an extra plot doesn't lead to more pagebreaks). This time it just asks for directions at three points; two are nullclines, and that is of course a point in itself to get them horizontal/vertical; it might be that some of them might even have brought sufficient notes or textbook bookmarks from examples in class, but they surely need to understand the question.

  Sets 2017 and 2020 were assigned for seminars. 2020 also had a similar connection between the differential equation system problem and a control problem.

How to solve:

(a)
- We are supposed to get an equation whose associated homogeneous equation is $\ddot{x} - \dot{x}\,\mathsf{tr}\,\mathbf{M} + x\,\mathsf{det}\,\mathbf{M} = 0$, but let us carry out the calculation:

$$\begin{aligned}\ddot{x} &= Q\dot{x} + R\dot{y} \\ &= Q\dot{x} + R\Big[Sx - Qy + 1\Big] \\ &= Q\dot{x} + RSx - Q[\dot{x} - Qx] + R \\ &= (Q^2 + RS)x + R\end{aligned}$$

- Differential equation becomes $\ddot{x} = 25x + 16$. e^{-5t} and e^{5t} are non-proportional solutions of the homogeneous, and then we have a constant particular solution $-16/(Q^2 + RS) = -16/25$. Solution: $\underline{\underline{C_1 e^{-5t} + C_2 e^{5t} - \frac{16}{25}}}$

(b) We need the Jacobian: $\mathbf{J} = \begin{pmatrix} \phi'(x) & 2y^{-3} \\ -y\phi''(x) & -\phi'(x) \end{pmatrix}$ which is of the form $\mathbf{M}$. Its determinant is $-\big(\phi'(x)\big)^2 + 2\phi''(x)y^{-2}$. At an equilibrium point where $\phi'' < 0$, this determinant is negative and thus we have a saddle point.

(c) We want an eigenvector $(v_1, v_2)'$ associated to the *negative* eigenvalue (and that eigenvalue is $-\sqrt{-\det \mathbf{J}}$ because trace is zero). The Jacobian is of the form $\mathbf{M}$, so we look for solutions of $(\mathbf{M} - \mu\mathbf{I})\mathbf{v} = \mathbf{0}$ i.e. $\begin{pmatrix} Q-\mu & R \\ S & -Q-\mu \end{pmatrix}\mathbf{v} = \mathbf{0}$. We have a superfluous equation, and $(R, \mu - Q)'$ solves the first equation and is an eigenvector (since $R = 2y^{-3} \neq 0$) and has slope $= (\mu - Q)/R$

We identify: $Q = \phi'(1) = 5 - 2x|_{x=1} = 3$, and $R = 2 \cdot (1/2){-3} = 2 \cdot 8 = 16$. Since $S = -\frac{1}{2} \cdot (-2) = 1$, we have the matrix from 1(b), where it gives the information that $\mu = -5$ is a negative eigenvalue. The slope is $\dfrac{-5-3}{16} = \underline{\underline{-\dfrac{1}{2}}}$.

(d) The U-shaped nullcline $y = \big((5-x)x\big)^{-1/2}$ is for x; the increasing curve has $\dot{y} = 0$.

A: <u>leftbound motion</u>; we are at the y-nullcline and below the x-nullcline; since $\frac{\partial}{\partial y}\big[(5-x)x - y^{-2}\big] = 2y^{-3}$ is > 0, «below» means $\dot{x} < 0$.

B: <u>downwards motion</u>; we are at the x-nullcline. Relative to the y-nullcline, we note that we are to the left, and $\frac{\partial}{\partial x}\big[\frac{3}{2} - y(5-2x) = 2y\big]$ is > 0, so «left of» means $\dot{y} < 0$.

C: <u>north-eastbound motion</u>: x-component has opposite sign of at A (reverse the argument to find what happens above the U-shaped curve) and y-component has opposite sign of at B (reverse that argument to find what happens to the right of the increasing curve).

Post-grading remarks: Part (a) averaged to «A». Part (b) not quite to «B». Part (c) was as predicted, demanding: most got failing score. Part (d) got «C» score. Some students might have thought that for points on nullclines, it was «intended» to show merely horizontal/vertical (not left/down), and to the extent they got (C) right, it was assumed they at least know it. Unfortunately, it is obvious that some think that «above nullcline» entails positive derivative, which is invalid (and in this case, gives wrong sign for point (B): it is above the $\dot{y}$-nullcline, but with downwards motion).

As pointed out for Problem 3, it was not too far from the phase diagram of a Ramsey–Cass–Koopmans model and a resource extraction problem covered (and exam 2020). However those had a vertical nullcline. A vertical line did show up in a couple of answers here too – not unlikely a combination of asymptote and memory. Some strange results followed.

Problem 3 † Let x_0 and T be positive constants. Consider the optimal control problem

$$\max_{u(t)\in[0,14]} \int_0^T \left(u(t) - \frac{3}{2}x(t)\right) dt \quad \text{when} \quad \dot{x} = (5-x)x - \frac{u^2}{4}, \quad x(0) = x_0, \quad x(T) \geq 0.$$

In the following, suppose that (x, u) satisfies the conditions from the maximum principle, with adjoint («costate») process $p = p(t)$.

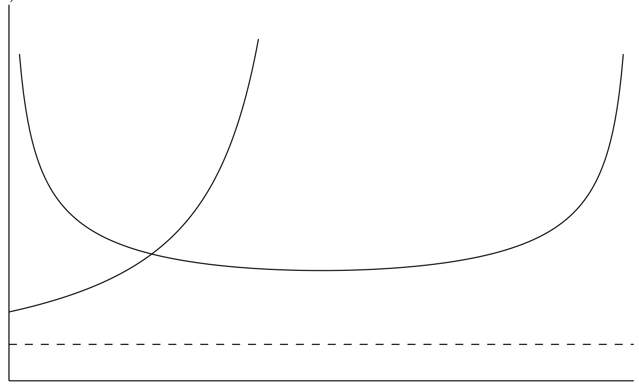
(a)
- State the conditions from the maximum principle.
- Could $u(t)$ be zero? (Warning: $p(t)$ *could* attain negative values.)

(b) Deduce the following differential equation system:

$$\dot{x} = (5-x)x - \left(\frac{1}{\max\{\,p\,,\,1/7\}}\right)^2 \qquad \text{i.e.} = \begin{cases} (5-x)x - 1/p^2 & \text{if } p \geq 1/7 \\ (5-x)x - 49 & \text{if } p < 1/7 \end{cases}$$

$$\dot{p} = \frac{3}{2} - (5-2x)p$$

(c) Take for granted that when $p \geq 1/7$, the system from (b) is the same as system (**) from Problem 2 part (c) with $y = p$, and that the line $p = 1/7$ lies below the nullclines like this (dashed):



- The conditions from the maximum principle restrict the point $(x(T), p(T))$ (that is, at final time T – the «endpoint» of the system!) to a «small» set in this diagram. Which set? I.e., *where could* $(x(T), p(T))$ *possibly be?*
- If furthermore $0 < u(t) < 14$ for all t large enough, $(x(T), p(T))$ is restricted further (i.e. we know more). *Where could* $(x(T), p(T))$ *possibly be in this case?*

Both questions ask for the whole «possible set» as follows from the conditions – not merely an example.

†Though it is surely possible to solve Problem 3 without Problem 2, it cannot be ruled out that some insight from Problem 2 could be useful.

Problem 3: Remarks This was constructed to be «not far from examples given in class and the last seminar», although when viewed as a control problem, it distinguishes itself from Ramsey–Cass–Koopmans and the resource extraction problem by having state-dependent running utility. It was deliberate to omit any discounting, as it was when something similar was given (with diagrams) in the 2020 exam, the one assigned for the last seminar.

- First bullet item of (a) is straightforward. One is free to include the dynamics for x.
- The second bullet item also serves as a hint that the maximality condition is not equivalent to a FOC. The conclusion might be useful when $0 < u < 14$ is assumed.
- (b): the second equation follows already from (a). The first is again a reminder that one can have endpoint solution, and with the «i.e. =» part it should not cause much problem. (At least not much «Math 3»-related problems – messing up considerations about the sign of p is not much related to the new theory.)
- Part (c) is «simply» about what to extract from the transversality condition.

  That has surely been covered – indeed it was pointed out in a finite-horizon Ramsey–Cass–Koopmans model that we would have to end up on an axis. And even more, since a student found a wrong inequality in an old exam note, I had to post an announcement after the lectures explaining it once again. However, pointing it out in a diagram could be slightly unfamiliar (that is, unfamiliar *in a Math 3 context* – I will not insult economics master-level students by making a more general claim that they are unfamiliar with identifying properties on a 2D diagram).

Problem 3: How to solve: We will need the Hamiltonian (which has no explicit «t»): $H(x,u,p) = u - \frac{3}{2}x + p\cdot(5-x)x - p\cdot\frac{1}{4}u^2$. Terms can be grouped as $u - \frac{p}{4}u^2 + \underbrace{px(5-x) - \frac{3}{2}x}_{\text{no «}u\text{»}}$.

The terms without u can be kept or (as here) dropped in the first condition of (a).

Furthermore they can in part (a) include or omit the differential equation for x. (It must hold, of course, but that doesn't mean they need to mention it – one condition set in the book does not.)

(a)
- Conditions:

  The optimal control maximizes $u - pu^2/4$ over $u \in [0, 14]$

  $\dot{p} = \frac{3}{2} - p\cdot(5-2x)$ with $p(T) \geq 0$ and if $x^*(T) > 0$: $p(T) = 0$.
- <u>No</u>; $\frac{\partial}{\partial u}\left[u - pu^2/4\right]\Big|_{u=0} = 1 - pu/2\Big|_{u=0} = 1$ is > 0, so we can always choose better than zero.

(b) We already have the second differential equation $\dot{p} = \frac{3}{2} - p\cdot(5-2x)$.

The first is deduced by inserting for the maximizing u:

If we have stationary maximum $u = 2/p$: it maximizes if $p > 0$ *and furthermore* it is admissible: $2/p \leq 14$. So if $p \geq 1/7$, then $u = 2/p$ and $u^2/4 = 4/(4p^2)$.

If p is not $\geq 1/7$, then $u - \frac{p}{4}u^2$ is increasing over the control region, and we choose the right endpoint $u = 14$. Then $u^2/4 = (2\cdot 7)^2/4 = 49$, verifying the other case.

(c)
- We have $x(T) \geq 0$ and $p(T) \geq 0$, so the point is in the first quadrant as in the diagram; but furthermore, one of these is zero. So $(x(T), p(T))$ is on an axis (more precisely, a positive half-axis – as picture restricts to).
- From (b): *stationary max, not at endpoint* – so $p > 1/7$. That means we must be on the second axis, above where it intersects the dashed $p = 1/7$ line.

Post-grading remarks: (a) and (b) averaged together, were «nearly B». Where to award points was sometimes not an exact science, as mistakes from (a) propagated to (b).

(a) First bullet item scored well except those who identified «maximizes» with «is a stationary point for». Second bullet item ... not so good. Even with the hint that p could be negative, there were claims about concavity, and uncritical use of FOC, like: making a blanket claim that $u = 2/p$ and since that could not be 0, ...

One paper claimed that yes, 0 is admissible. True statement, but not answering a question about a control satisfying the conditions.

(b) Split formulae are hard, apparently ... all sorts of mistakes were seen here. Working as if $u \mapsto H$ were convex and checking only endpoints; claiming that since $u = 2/p$ must be in $[0, 14]$ then p simply cannot be too small. (In addition to getting the logic wrong, this does also violate the transversality condition x_0 is so big (or time is so short) that running at $u \equiv 14$ won't x down to zero within the horizon.)

(c) Nobody did well on this. A couple did well on the first bullet item. Scores were calculated with reduced weight.

Problem 4 Let $a > 0$ and $p \in (0,1)$ be constants. Consider the dynamic programming problem

$$J_{t_0}(x) = \max_{u_t \in [0,1]} \left\{ x_T + a^{1-p} \cdot x_T^p + \sum_{t=t_0}^{T-1} \left(u_t x_t + (u_t x_t)^p \right) \right\}$$

$$\text{where } x_{t+1} = x_t - u_t x_t, \quad x_{t_0} = x \ (> 0)$$

‡It is possible (and could maybe be quicker) to show (b) first – possibly even with the hint for the more general «bonus» case – and then deduce the answer to (a) from there.

(a) Find $J_{T-2}(x)$ if $a = p = 2/3$.

(b) In this part, allow for more general constants: any $p \in (0,1)$ and any $a > 0$.

- Show that at time $t = T - k$, we have the form

$$J_{T-k}(x) = x + A_k^{1-p} \cdot x^p \quad \text{where } A_k \text{ does not depend on } x,$$

- Deduce a difference equation for A_k, starting at $A_0 = a$.
  (Carry out the maximization to needed to eliminate the «u» letter. Answer will look fairly simple in the end.)

«*Hint and possible bonus*§» How far can you get in the first bullet item if you allow for the following generalizations? (I): The number p might either be $\in (0,1)$ or > 1, and (II): In place of $u_t \in [0,1]$, assume $u_t \in U$ where U is closed and bounded and contained in $[0,1]$

‡The assumptions ensure that $x_t \geq 0$ (and $x_{t-1} \geq x_t$) for all $t = t_0 + 1, \dots, T$, so $(u_t x_t)^p$ is well-defined. a^{1-p} means a to the $(1-p)$th power and is just another constant. The form with the $(1-p)$th power, also occurring in A_k^{1-p} in part (b), might hopefully make some calculations more convenient.

§For (b):

- As «hint»: Depending on your workflow, this could speed up some of your calculations.
- «possible bonus» means that everything but this correct, will be sufficient for 100 percent score – but that the graders *may* at their discretion, take an insightful (brief!) observation here into account on papers that would otherwise be «narrowly missing a grade».

Problem 4: general remark It was deliberate to give a problem very similar to one assigned for Seminar 3. That one had $a = 1$, $p = 1/2$, but a discounting parameter – which in calculations was «offset» by how the same parameter occurred in the dynamics.

Problem 4: how to solve (doing part (b) first, and with the «bonus» hint.)

The stated form holds initially with $A_0 = a > 0$. Assume for induction that the stated form holds true at k, and with $A_k > 0$. Then at $k+1$, we get value (with «A» for A_k):

$$\max_{u\in U}\left\{ux + (ux)^p + (x-ux) + A^{1-p}\cdot(x-ux)^p\right\} = x + x^p \cdot \underbrace{\max_{u\in U}\left\{u^p + A^{1-p}\cdot(1-u)^p\right\}}_{=A_{k+1}^{1-p},\text{ is }>0,\text{ no «}x\text{»}}$$

where the extreme value theorem grants existence of the maximum over a closed and bounded U of a continuous function. *This proves the form,* thus answering the first bullet item. [Which was the reason for writing that «bonus hint» into the problem set.]

For the difference equation, we need to specialize. With $p \in (0,1)$ and $A > 0$, we are maximizing over a concave which has vertical tangents at $u = 0$ and at $u = 1$, so there will be a stationary maximum where $u^{p-1} = A^{1-p}(1-u)^{p-1}$ i.e. $Au = 1-u$, so that $u = 1/(1+A)$ and $1-u = A/(1+A)$. Inserting:

$$\frac{1}{(1+A)^p} + A^{1-p}\Big(\frac{A}{1+A}\Big)^p = (1+A)^{-p}\cdot\left[1 + A^{1-p+p}\right] = (1+A)^{1-p}$$

So $A_{k+1}^{1-p} = (1+A_k)^{1-p}$ leading to $\underline{\underline{A_{k+1} = 1 + A_k}}$.

From this, we can answer part (a) as a corollary: $A_2 = 2 + A_0 = 8/3$ so that

$$J_{T-2}(x) = x + x^{2/3}\cdot(8/3)^{1-2/3} = \underline{\underline{x + 2x^{2/3}\cdot 3^{-1/3}}}$$

Post-grading remarks: This was no fun to grade. Single-variable maximization messed up big time. The concavity of root functions should be known, but there were several claims of solution at $u = 0$. Let us differentiate $(ux)^p + A^{1-p}(x-ux)^p$ wrt. u to get $p(ux)^{p-1}\cdot x + pA^{1-p}(x-ux)^{p-1}\cdot(-x)$. Various papers would forget/omit the red x, the blue $-x$, the $p-1$ exponent or the green -1. (Or one of the p's in front, but that wouldn't make for same level of disaster.)

Part (b) must be graded taking into account that posing and carrying out and induction proof is a large part of what is intended to be tested. Unfortunately, errors like the above make it hard to assess whether they realize they are supposed to end up with what is to be proven.

Problem 5 Let f be a function of a real variable, and *continuous* on the entire $\mathbf{R}$. Show that if f is one-to-one, i.e. it has an inverse function f^{-1}, then f^{-1} is quasiconcave.

Hints (if you need them):

- You *will* need a particular property that follows from continuity.[¶] You are allowed to assume continuous differentiability if that makes it simpler, but that property is not essential.
- Recall that an inverse function means $f^{-1}(f(x)) = x$ (all $x \in \mathbf{R}$) and $f(f^{-1}(y)) = y$ (all y in the range of f) – we are not talking about one divided by anything.

Problem 5, how to solve and remarks: The following is a complete answer:

> «A continuous one-to-one function on an interval is strictly monotone, thus so is its inverse, and a monotone function on an interval is quasiconcave.»

- Here the graders must expect some handwaving, as inverse functions haven't been treated with «Math 3 level rigor».

  (Allowing the C^1 condition was to facilitate an argument of the form that $f^{-1}(f(x)) = x$ with derivative $= 1$, so that derivatives $(f^{-1})'(y) \cdot f'(x)$ cannot change sign; it was intentionally opaque on precisely what could be assumed C^1, and it is then perfectly fine to tacitly assume f^{-1} to be C^1, necessitating $f' \neq 0$.)
- Once monotonicity is established, then one *can* of course argue in more formulae, although it isn't needed:
  $f^{-1}(x) \geq L$ is equivalent with either $x \geq f(L)$ (strictly increasing case) or $x \leq f(L)$ (strictly decreasing case), and sets $[f(L), \infty)$ and $(-\infty, f(L)]$ are both convex.

Post-grading remarks: Most positive thing to say is that somebody did it correctly. (And somebody nearly did, but crossed it out and wrote something irrelevant instead). One may ask whether *inverse functions* is a topic completely forgotten by now? Differentiating? Nobody did. But seriously, continuity in itself is not sufficient for quasiconcavity.

Scores were calculated with lower weight on this problem, for those papers who did not cover it adequately.

[¶] Indeed if you drop the assumption that f is continuous on its (convex!) domain, the claim is false! Counterexamples without that assumption: $g(t) = 1/t$ defined except at 0: it is its own inverse, but not a quasiconcave function. h defined as $h(t) = g(t)$ and $h(0) = 0$ (thus defined on $\mathbf{R}$) is also its own inverse, but not continuous at 0 and not quasiconcave.