University of Oslo / Department of Economics / NCF

ECON4140 Mathematics 3: the 2020-06-23 exam solved

See the ordinary exams' grading guidelines for considerations concerning ECON4140 exams in general and spring 2020 in particular.

Problem 1 of 5. Consider the following differential equation and difference equation:

$$\ddot{x} = \tfrac{1}{2}\dot{x} - \tfrac{1}{8}x + f(t) \qquad \text{(D)}$$
$$x_{t+2} = \tfrac{1}{2}x_{t+1} - \tfrac{1}{8}x_t + g_t \qquad \text{(E)}$$

(a) Find the general solutions of (D) and (E) when $f(t) = 2020 = g_t$ (constants).

(b) Explain how you would go forth to solve (D) when $f(t) = 2020+t^{2020}+2020\sin(2020\,t)$.

(c)
- Is (D) stable? Is (E) stable?
- Do we have tools to decide stability of (D) and/or (E) *without* solving?

Solution: (c) first:

(c) With $a = -1/2$ and $b = 1/8$, we have $\underline{\text{(D) unstable}}$ (as $a < 0$) and $\underline{\underline{\text{(E) stable}}}$ (because both $1/2 = |a|$ is $< 1 + b$ and $b = 1/8$ is < 1).
[Notes: calculating the characteristic roots $(1 \pm i)/4$ first, shall qualify as «without solving» although one is quite near a solution by then. Moreover, it is not asked for asymptotic stability nor for the «outwards spiraling» nature of the unstable solution]

(a) The characteristic equation $r^2 - r/2 + 1/8 = 0$ yields $2r = 1/2 \pm \sqrt{1/4 - 1/2}$ and non-real roots $r = (1 \pm i)/4$.

(D): Constant particular solution satisfying $0 = -u^*/8 = 2020$, so general solution $\underline{\underline{16160 + e^{t/4}\big(C\cos(t/4) + D\sin(t/4)\big)}}$.

(E): Constant particular solution satisfying $u^* \cdot [1 - 1/2 + 1/8] = 2020 \Leftrightarrow u^* = 16160/5 = 3232$. General solution:
$\underline{\underline{3232 + e^{t/\sqrt{8}}\big(C\cos(\theta t) + D\sin(\theta t)\big)}}$ with $\cos\theta = \frac{-1/2}{2/\sqrt{8}} = -\frac{2\sqrt{2}}{4} = \underline{\underline{-\frac{1}{2}\sqrt{2}}}$.

[Note: it is OK to stop here (or $\cos\theta = \sqrt{1/2}$ by symmetry); it is not required to get to $\theta = -\pi/4$ (or $\pi/4$), as the course has focused on the recipe and by no means on trigonometric tables.]

(b) Try $u^* = K\cos(2020t) + L\sin(2020t) + q(t)$ where q is a 2020-degree polynomial, and fit all 2023 coefficients. [Note the essential point: both a $\cos$ and a $\sin$ are needed, and a full polynomial. Not merely «$\sin$» and «$Qt^{2020} + R$».]

Problem 2 of 5. Consider the differential equation system

$$\begin{aligned} \dot{x} &= y - \cos\frac{\pi x}{3} - \frac{1}{2} \\ \dot{y} &= |x| - y \end{aligned} \tag{S}$$

(a) The system has an equilibrium point (a.k.a. «stationary state») at $(x, y) = (1, 1)$. Show that it is a saddle point.

(b) The system has one more equilibrium point $(\tilde{x}, \tilde{y})$. Find and classify it:

- Decide if it is stable or not;
- Decide if it is oscillating or not;
- If unstable, decide if it is a saddle point.

(c) There is a non-constant particular solution $(x(t), y(t))$ which converges to $(1, 1)$ as $t \to +\infty$. Find $\lim\limits_{t\to+\infty} \dfrac{y(t)-1}{x(t)-1}$ for this solution path.

Solution: For (a) and (b) we need the Jacobian matrix $\begin{pmatrix} \frac{\pi}{3}\sin\frac{\pi x}{3} & 1 \\ \operatorname{sign} x & -1 \end{pmatrix}$

(a) At $(1, 1)$, the Jacobian becomes $\begin{pmatrix} \frac{\pi}{3}\sin\frac{\pi}{3} & 1 \\ 1 & -1 \end{pmatrix}$. The determinant $-(\frac{\pi}{3}\sin\frac{\pi}{3} + 1)$ is *negative* (as $\sin(\pi/3) > 0$).

(b) Both cosine and absolute value are even functions, so because $(1, 1)$ is an equilibrium point, so is $(-1, 1)$. There, the Jacobian has determinant $1 - \frac{\pi}{3}\sin\frac{-\pi}{3} = 1 + \frac{\pi}{3}\sin\frac{\pi}{3} > 0$. The trace is negative as $\frac{\pi}{3}\sin\frac{\pi}{3} < 1$. So it is stable. It is spiralling, i.e. <u>dampened oscillations</u> if $\mathsf{tr}^2 - 4\,\mathsf{det}$ is negative, which is the case: for $\mathbf{A} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, we have $(a-d)^2 + 4bc$ which becomes $(1 - \frac{\pi}{3}\sin\frac{\pi}{3})^2 - 4$ which is < 0. [Notes: With a calculator at hand, one must arguably accept the claim that $(1 - \frac{\pi}{3}\sin\frac{\pi}{3})^2 - 4 < 0$ without further argument. (It is probably not the best use of exam time to elaborate on it.)]

(c) The limit is the slope of the eigenvector corresponding to the *negative* eigenvalue μ of the Jacobian at $(1, 1)$. That eigenvector satisfies $(1\,,\,-1-\mu)\mathbf{v} = 0$, and so $\mathbf{v} = (1+\mu\,,\,1)'$ has slope $1 + \mu = \underline{\underline{1 + \frac{1}{2}\Big(\frac{\pi}{3} - 1 - \sqrt{(\frac{\pi}{3} - 1)^2 + 4\frac{\pi}{3} + 4}\Big)}}$.

[Note: The «mock exam» only asked for the sign of the slope, as that was easier in that problem. Here, it is arguably easier to calculate the slope first, hence the question formulation. Getting the sign would likely in practice have become a calculator exercise, rather than rewriting into $\frac{1}{2}\big(\frac{\pi}{3} + 1 - \sqrt{(\frac{\pi}{3}+1)^2 + 4}\big)$.]

Problem 3 of 5. Consider the variational problem(s)

$$\max / \min \int_0^T e^{-rt} \ln \Big(G(x(t), \dot{x}(t)) \Big)\, dt \quad \text{subject to} \quad x(0) = x_0, \quad x(T) = 0$$

where all constants are > 0. This problem concerns the function

$$G(x, y) = R + (2 - x) \cdot x - y, \qquad R > 0 \text{ constant}$$

but for part (a) it is probably a good idea – and it is worth partial score by itself – to first use a general G (but insert for $\partial G/\partial y = -1$).

(a) Write out the associated Euler equation.

(b) Suppose we have found an $x = x(t)$ that satisfies the Euler equation with $x(0) = x_0$ and $x(T) = 0$. Can we then conclude that this solves the maximization problem? The minimization problem? Neither?

Solution: (b) does not need (a):

(b) We have G a sum of concave functions, and then $\ln G$ is an increasing concave transformation and $e^{-rt} > 0$. So the integrand is concave in $(x, \dot{x})$ for each t and we will have a $\underline{\underline{\text{solution of the maximum problem}}}$.

(a) To calculate $\frac{\partial F}{\partial x} - \frac{d}{dt}\frac{\partial F}{\partial \dot{x}}$, we write as $e^{-rt}\frac{1}{G}\frac{\partial G}{\partial x} - \frac{d}{dt}\left[e^{-rt}\frac{1}{G}\frac{\partial G}{\partial y}\right] = e^{-rt}\frac{2-2x}{G} + \frac{d}{dt}\left[e^{-rt}\frac{1}{G}\right]$ and the latter term is $-re^{-rt}/G - e^{-rt}/G^2 \cdot \frac{d}{dt}G(x(t), \dot{x}(t))$. Multiplying by $G^2 e^{rt}$ and inserting:

$$(2 - 2x)G = rG + \frac{\partial G}{\partial x}\dot{x} + \frac{\partial G}{\partial y}\ddot{x} = rG + (2 - 2x)\dot{x} - \ddot{x}$$

so that $\quad \underline{\underline{\ddot{x} = r \cdot \big(R + (2 - x)x - \dot{x}\big) + 2(1 - x) \cdot \big(2\dot{x} - R - (2 - x)x\big)}}$

Problem 4 of 5. Let $\beta \in (0,1)$ be a constant. Consider the dynamic programming problem

$$J_t(x_t) = \max_{u_t \in U} \sum_{s=t}^{T} \beta^{s-t}(x_t - \beta u_t^2) \qquad \text{subject to} \quad x_{t+1} = \beta x_t + (1-u_t)\cdot u_t$$

You can choose – once for the entire Problem 4 – whether to use $U = (-\infty, +\infty)$ or $[0,\infty)$ or $[0,1]$. Your choice might affect what arguments you need to complete part (a).

In the following, neither A_τ nor B_τ (nor a_τ nor b_τ) can depend on x, only on time τ *remaining* to the end of the planning horizon.

(a) Show by induction that the value $J_{T-\tau}(x)$ takes the form $A_\tau x + B_\tau$, and find a difference equation for A_τ.

If you prefer not to use the β^{s-t} formulation, you can equivalently show that the function $M_t = \beta^t J_t(x_t) = \max_{u_t \geq 0} \sum_{s=t}^{T} \beta^s(x_t - \beta u_t^2)$, takes the form $M_{T-\tau} = a_\tau x + b_\tau$; then find difference equations for a_τ and b_τ.

(b) Put $t = 0$. Let $T \to +\infty$ to get an infinite-horizon problem.

- State the associated Bellman equation for the value function $J(x)$.

  (In this problem you must use J.)

- Show that if $J(x) = Ax + B$ satisfies the Bellman equation (with A and B constants) then $A = 1/(1-\beta^2)$.

  (You are not asked to show that such a function actually solves the infinite horizon problem.)

Solution (for all three choices of U):

(a) True at $\tau = 0$: $J_T(x) = \max_{u\in U}\{x - \beta u^2\} = x$ with $u = 0$, all three choices of U.

Suppose true at τ. Then at $\tau + 1$:

$$\begin{aligned} J_{T-(\tau+1)}(x) &= \max_{u\in U}\Big\{x - \beta u^2 + \beta J_{T-\tau}(\beta x + (1-u)u)\Big\} \\ &= \max_{u\in U}\Big\{x - \beta u^2 + \beta A_\tau \cdot \big[\beta x + (1-u)u + B_\tau\big]\Big\} \\ &= \underbrace{(1+\beta^2 A_\tau)}_{=A_{\tau+1}} x + \underbrace{\beta B_\tau + \beta \max_{u\in U}\Big\{-u^2 + A_\tau\cdot(1-u)u\Big\}}_{=B_{\tau+1},\text{ does not depend on } x} \end{aligned}$$

provided the maximum exists; it will for U compact, by the extreme value theorem, but for all three alternatives it suffices that we can show that the second-order coefficient $-(1+A_\tau)$ is negative, as then we have a concave *quadratic*. And $\underline{\underline{A_{\tau+1} = 1 + \beta^2 A_\tau}}$ is positive by induction, as $A_0 = 1 > 0$ and $\beta > 0$.

(b) The Bellman equation becomes as the dynamic programming equation except dropping the time indices: $\underline{\underline{J(x) = \max_{u\in U}\Big\{x - \beta u^2 + \beta J(\beta x + (1-u)u)\Big\}}}$.

Testing the form $Ax + B$, we insert to get:

$$\begin{aligned} Ax + B &= \max_{u\in U}\Big\{x - \beta u^2 + \beta A \cdot \big[\beta x + (1-u)u)\big] + \beta B\Big\} \\ &= (1+\beta^2 A)x + \beta B + \beta \max_{u\in U}\Big\{-u^2 + A(1-u)u\Big\} \end{aligned}$$

and we match x coefficients: $(1-\beta^2)Ax = x$ and so $A = \frac{1}{1-\beta^2}$.

Problem 5 of 5. Let $n \geq 2$ be a natural number and s be a real constant. Define $\mathbf{J}_n$ to be the $n \times n$ matrix where element (k, ℓ) is 1 if $k + \ell = n + 1$, and 0 otherwise[1]. Note that $\mathbf{J}_3$ has a «middle element» number $(2, 2)$, while $\mathbf{J}_4$ has no single middle element.

Let the matrix $\mathbf{A} = \mathbf{A}_{n,s}$ be defined as $\mathbf{I}_n + s\mathbf{J}_n$, where $\mathbf{I}_n$ is the $n \times n$ identity matrix.

(a) Let $\mathbf{a} = (1, 0, \ldots, 0)'$ and $\mathbf{b} = (0, \ldots, 0, 1)'$ have only a single nonzero element. Let $\mathbf{u} = \mathbf{a} + \mathbf{b}$ and $\mathbf{v} = \mathbf{a} - \mathbf{b}$.

For each n and each s: Check whether the vectors $\mathbf{u}$ and/or $\mathbf{v}$ is/are eigenvector for $\mathbf{J}_n$ and/or $\mathbf{A}_{n,s}$.

(b) In this part let $n = 3$ and $s \neq 0$.

- Calculate the characteristic polynomial of $\mathbf{A}$.
- Find – or disprove the existence of – an eigenvector $\mathbf{w}$ which is *not* a linear combination of $\mathbf{a}$ and $\mathbf{b}$.

(c) Suppose that the rank $\mathbf{A}_{n,1}$ equals n. Can then s be equal to 1? Your answer might depend on n.

(d) Let $\mathbf{Y}$ and $\mathbf{Z}$ be symmetric $n \times n$ matrices, both of which have 0 as their smallest eigenvalue. Prove the following facts about $\mathbf{M} = \mathbf{Y} + \mathbf{Z}$:

- $\mathbf{M}$ has no negative eigenvalues.
- [Might be difficult.] If $\mathbf{Y}$ and $\mathbf{Z}$ have *no eigenvector in common,* we know that $\mathbf{M}$ has all its eigenvalues positive.

Solution:

(a) Left-multiplying by the matrix $\mathbf{J}$ turns every vector upside-down. That does not change $\mathbf{u}$, but it changes sign on $\mathbf{v}$. We have:

u: We have $\mathbf{Ju} = \mathbf{u}$ (eigenvector!) and $\mathbf{Au} = \mathbf{Iu} + s\mathbf{Ju} = (1+s)\mathbf{u}$ (eigenvector!).

v: Now, $\mathbf{Jv} = -\mathbf{v}$ (eigenvector!) and $\mathbf{Av} = \mathbf{Iv} + s\mathbf{Jv} = (1-s)\mathbf{v}$ (eigenvector!).

(b)
- The characteristic polynomial: $|\mathbf{A} - \mu\mathbf{I}| = \begin{vmatrix} 1-\mu & 0 & s \\ 0 & 1+s-\mu & 0 \\ s & 0 & 1-\mu \end{vmatrix} = (1-\mu)\begin{vmatrix} 1+s-\mu & 0 \\ 0 & 1-\mu \end{vmatrix} + s\begin{vmatrix} 0 & 1+s-\mu \\ s & 0 \end{vmatrix} = \underline{\underline{[(1-\mu)^2 - s^2](1+s-\mu)}}.$

[1]Examples: $\mathbf{J}_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\mathbf{J}_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ while $\mathbf{J}_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

- $\mu = 1 + s$ is a double root, so by (a) any third eigenvector must correspond to this. $\mathbf{A} - (1+s)\mathbf{I} = \mathbf{I} + s\mathbf{J} - \mathbf{I} - s\mathbf{I} = s(\mathbf{J} - \mathbf{I})$ and because $s \neq 0$, $\mathbf{w}$ must satisfy $\mathbf{Jw} = \mathbf{w}$, i.e. be itself turned upside down. That is, $\underline{\underline{\mathbf{w} = (0\,,\,1\,,\,0)'}}$.

  [The answer is not unique. Any linear combination of $\mathbf{u}$ and $(0, 1, 0)'$ is an eigenvector, and an equivalent answer could be e.g. $(1, 1, 1)'$.]

(c) No. When $s = 1$, the first and last columns are equal.
[Alternatively: when $s = 1$, we have an eigenvalue for $1 - s = 0$, and so the determinant vanishes. This question really only checks whether one knows what it takes to have full rank.]

(d)
- With all eigenvalues nonnegative, $\mathbf{Y}$ and $\mathbf{Z}$ are both positive semidefinite, and so $\mathbf{x}'(\mathbf{Y} + \mathbf{Z})\mathbf{x} = \mathbf{x}'\mathbf{Y}\mathbf{x} + \mathbf{x}'\mathbf{Z}\mathbf{x} \geq 0$. $\mathbf{M}$ is positive semidefinte and therefore has all eigenvalues nonnegative.
- Suppose for contradiction that there is an $\mathbf{x} \neq \mathbf{0}$ such that $\mathbf{x}'\mathbf{Y}\mathbf{x} + \mathbf{x}'\mathbf{Z}\mathbf{x} = 0$, and as both terms are nonnegative, we must have $\mathbf{x}'\mathbf{Y}\mathbf{x} = 0$ and $\mathbf{x}'\mathbf{Z}\mathbf{x} = 0$. By positive semidefiniteness, 0 is the minimum value.

  The Lagrange first-order condition for max/min of $\mathbf{x}'\mathbf{Y}\mathbf{x}$ subject to $||\mathbf{x}|| = 1$, is that $\mathbf{x}$ is an eigenvector of $(\mathbf{Y} + \mathbf{Y}')/2 = \mathbf{Y}$. Similarly, $\mathbf{x}$ must be an eigenvector of $\mathbf{Z}$. This contradicts the assumption that we have no common eigenvectors.