Lecture 2

The "maximal number of linearly independent vectors in S":

k if there exists a linearly independent set of k vectors from S, but no such set of $k+1$.
($k = 0$ if $S = \{\vec{0}\}$)

Def. The rank $r(\vec{A})$ of a matrix $\vec{A}$, is the maximal number of linearly independent column vectors of $\vec{A}$.

Fact: equals the max # of lin. indep row vectors of $\vec{A}$

and

equals the "order" of the largest nonzero minor of $\vec{A}$

k if the minor is $k \times k$

(with $r(\vec{0}) = 0$.)

The first fact follows from the second, which indicates how to calculate.

An $m \times n$ matrix has rank $\leq \min\{m,n\}$.

If the rank equals $\min\{m,n\}$: "full rank".

(Otherwise : "rank-deficient".)

(If you ever see e.g. "full row rank", it means $r(\vec{A}) = m \leq n$)

Example 1: $\vec{A} = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}$ has rank equal to $\underline{2}$.

< 3 because the only 3×3 minor (namely $|\vec{A}|$) is zero

≥ 2 because $\begin{vmatrix} 1 & 4 \\ 2 & 5 \end{vmatrix} \neq 0$. (for "$\geq 2$" it suffices that one 2×2 minor is $\neq 0$)

Example 2: $\vec{A} = \begin{pmatrix} 11 & 21 & 31 & \cdots & 91 \\ 12 & 22 & 32 & \cdots & 92 \\ 13 & 23 & 33 & \cdots & 93 \\ 14 & 24 & 34 & \cdots & 94 \end{pmatrix}$

has rank two as well.

Err... you don't want to calculate all 4×4 and all 3×3 minors to find out...?

Example 2 cont'd: how to calculate (by hand)?

Use elementary row or column operations!

Recall: $\begin{vmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 6 \\ 2 & 3 & 6 \\ 3 & 3 & 6 \end{vmatrix} = 0.$

$-1 \quad\quad -2$

Do this on the big rectangular matrix:

$$r\begin{pmatrix} 11 & 21 & \cdots & 91 \\ 12 & 22 & & 92 \\ 13 & 23 & & 93 \\ 14 & 24 & \cdots & 94 \end{pmatrix} = r\begin{pmatrix} 11 & 10 & \overbrace{20 \quad \cdots \quad 80}^{\text{can be deleted...}} \\ 12 & 10 & 20 \quad\quad\quad 80 \\ 13 & 10 & 20 \quad\quad\quad 80 \\ 14 & 10 & 20 \quad \cdots \quad 80 \end{pmatrix}$$

-1

... because: -2, -3, ..., -8

$$= r\begin{pmatrix} 11 & 1 \\ 12 & 1 \\ 13 & 1 \\ 14 & 1 \end{pmatrix} \begin{cases} \leq 2, \text{ since } 2 \text{ columns} \\ \geq 2, \text{ since } \begin{vmatrix} 11 & 1 \\ 12 & 1 \end{vmatrix} \neq 0. \end{cases}$$

why?

This approach is also useful for questions like: "Is $S = \{ \ldots \}$ linearly dependent?"

Example 3: [...]

Exam Q could be: Let $\vec{A}_t = ($[depends on t]$)$.

Find the rank, for each $t \in \mathbb{R}$.

Rank and linear eq. systems $\vec{A}\,\vec{X} = \vec{B}$

(Important!) fact: we have the equivalence

$\vec{A}\,\vec{X} = \vec{B}$ has solution

I if and only if

$r(\vec{A}) = r(\underbrace{\vec{A} \,\vdots\, \vec{B}})$

the augmented coeff. matrix.

In II and III, assume $\vec{B} = \vec{b}$ is a vector.

Assume $r(\vec{A}) = r(\vec{A} \,\vdots\, \vec{b}) = r$

so a solution exists.

II Let $\vec{A}$ be $m \times n$. Then

* there are $n - r$ degrees of freedom
* there are $m - r$ superfluous eq's.

But there is more. Assume we do have solution, $r(\vec{A}) = r(\vec{A} \,\vdots\, \vec{b}) = r$.

III
- Let an $r \times r$ submatrix $\vec{M}$ of $\vec{A}$ ← not of $(\vec{A} \,\vdots\, \vec{b})$! have $|\vec{M}| \neq 0$. (that is, $|\vec{M}|$ is one $r \times r$ minor).
- Mark off the elements of $\vec{M}$ in $\vec{A}$. Then:
  - → the "other" rows (not intersecting $\vec{M}$) can be deleted
  - → the "other" col's correspond to x_i we can choose freely.

Why (I)? You don't need proof, but you should catch what goes on:

- $\vec{A}\vec{x} = x_1 \vec{a}^{(1)} + \ldots + x_n \vec{a}^{(n)}$
  ($\vec{a}^{(i)}$ columns of $\vec{A}$)

- If some $a^{(i)}$ can be written as linear combination of the others, then do that
  [eliminating a degree of freedom, if there is solution!]
  Repeat until r linearly indep. col's remain.

- If $\Sigma x_i \vec{a}^{(i)} = \vec{b}$ then $\vec{b}$ can be written as a linear combination.
  [Fancy math speak: $\vec{b} \in$ the span of the col's]

So augmenting with $\vec{b}$ cannot increase the # of lin. indep. vectors if there is a sol'n; conversely, if it does not do so, we do have a solution.

- Solution $\Leftrightarrow$ $\vec{A}$ and $(\vec{A} : \vec{b})$ have same # of lin. indep. col's $\Leftrightarrow$ same rank.

- For $\vec{A}\overleftrightarrow{X} = \vec{B}$, this must hold for $\vec{A}$ vs $(\vec{A} : \vec{b}_i)$, every column $\vec{b}_i$ of $\vec{B}$.

Why II and III ?

Loosely:

- # of lin. indep rows = # lin. indep col's.
- rows — i.e. eq's — that can be written in terms of the others, can be deleted.
- An $r \times r$ minor has the largest possible number of "lin. indep left-hand sides" and thus also "right-hand sides" since we assume we have solution!

Delete these superfluous eq's.

- Left with r eq's in n unknowns.
- If after "moving $n-r$ var's to the RHS" we have an invertible eq. system, then no matter how we choose those $n-r$, we have unique values for the rest.

Picture: If $r(\vec{A}) = r(\vec{A} : \vec{b})$: and, e.g

$$\vec{A} = \begin{pmatrix} \cdot & \odot & \cdot & \cdot & \odot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \odot & \cdot & \cdot & \odot & \cdot \end{pmatrix}$$

where the circled elements form a largest nonzero minor, then we have 4 deg's of freedom and

- we can delete eq #2
- we can choose x_1, x_3, x_4, x_6 freely.

Example: Consider
$$\begin{pmatrix} 1 & 6 & 2 & 1 \\ -2 & -5 & 0 & -1 \\ 3 & 4 & p & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ q \\ 2 \end{pmatrix}$$

For each p, q, decide the number of solutions / degrees of freedom. [Can there possibly be unique sol'n?]

- $r(\vec{A}) \geq 2$ (e.g., $\begin{vmatrix} 2 & 1 \\ 0 & -1 \end{vmatrix} \neq 0$.)
- $\vec{a}^{(4)} = \frac{1}{7}\left(\vec{a}^{(1)} + \vec{a}^{(2)}\right)$ so

$$r(\vec{A}) = r\underbrace{\begin{pmatrix} 1 & 6 & 2 \\ -2 & -5 & 0 \\ 3 & 4 & p \end{pmatrix}} = \begin{cases} 3 & \text{for } p \neq -2 \\ 2 & \text{for } p = -2 \end{cases}$$

has determinant $7p + 14$

Solution with 1 d.f. (x_4 can be free) for $p \neq -2$.

Let $p = -2$ $\quad r(\vec{A} \,\vdots\, \vec{b}) = r(\vec{a}^{(1)} \,\vdots\, \vec{a}^{(2)} \,\vdots\, \vec{b})$ (WHY?)

$$= r\begin{pmatrix} 1 & 6 & 1 \\ -2 & -5 & q \\ 3 & 4 & 2 \end{pmatrix} = r\begin{pmatrix} 1 & 6 & 1 \\ -2 & -5 & q \\ 0 & -13 & q \end{pmatrix}$$

determinant $7q + 13(q+2) = 20q + 26$.

So for $p = -2$, $q \neq -\frac{13}{10}$: $\underline{\text{no solution}}$

For $p = -2$, $q = -\frac{13}{10}$: $r(\vec{A}) = r(\vec{A} \,\vdots\, \vec{b}) = 2$

and solution with two degrees of freedom (e.g. x_3 and x_4)

Example:

Does
$$\begin{pmatrix} 9 & 41 \\ 8 & 42 \\ 7 & 43 \\ \vdots & \vdots \\ 1 & 49 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$
have

no, one or $\underbrace{\text{infinitely}}$ many solutions?

if so: only one degree of freedom! (WHY?)

Non-prop col's, so $r(\vec{A}) = 2$. (what can we say already?)

$$r(\vec{A} \,;\, \vec{b}) = r\begin{pmatrix} 9 & 41 & 1 \\ \vdots & \vdots & \vdots \\ 1 & 49 & 1 \end{pmatrix} = 2$$

+1 second col becomes

50. third

So the eq. system is $\Leftrightarrow$ to

$$\begin{pmatrix} 9 & 41 \\ 8 & 42 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$, unique sol'n.