Lecture 3

Eigenvalues & eigenvectors

[This is both an exam topic in its own right, and something used for other topics!]

The big Q: When is $\vec{A}\vec{x} = \lambda\vec{x}$ (*)
for some $\vec{x} \neq \vec{0}$ and some number λ?

Def. Fix a matrix $\vec{A}$, (necessarily square).
If for some $\vec{x} \neq \vec{0}$ there exists a number λ such that $\vec{A}\vec{x} = \lambda\vec{x}$, we say that
$\vec{x}$ is an eigenvector of $\vec{A}$ with corresponding eigenvalue λ

Why bother?

Use #1: We shall cover the following facts in this course: Let $f \in C^2(\mathbb{R}^n)$. We have
the Hessian has all eigenvalues positive everywhere
$\Rightarrow$ f strictly convex $\Rightarrow$ f convex
$\Rightarrow$ the Hessian has all eigenvalues nonnegative everywhere.

Use #. 2 Let $\vec{x} = \vec{x}(t)$ follow the differential eq.

constant matrix.

$$\frac{d}{dt} \vec{x}(t) = \vec{A} \vec{x}(t)$$

If some eigenvalue of $\vec{A}$ is >0, then some particular solutions diverge (instability)

Use # 3: See this link to BI's "Math 2" course, pp 1.-8.

http://www.dr-eriksen.no/teaching/GRA6035/2010/lecture3-hand.pdf

(We will <u>not</u> cover his slides 17-18, and barely touch his pp 20 ff.)

$\vec{A}\vec{x} = \lambda \vec{x}$ – i.e., $(\vec{A} - \lambda \vec{I})\vec{x} = \vec{0}$.

General remarks:

- $\vec{0}$ is <u>not</u> an eigenvector. We are asking for the nontrivial ones.
- If $\vec{v}$ is an eigenvalue, then so is $t\vec{v}$, any $t \neq 0$. These are often "identified as one": the phrase
  "k eigenvectors" usually means
  "k linearly independent eigenvectors".
  (max n, if $\vec{A}$ is $n \times n$. Why?)

(a) If asked something like:
Show that $\begin{pmatrix}1\\2\\1\end{pmatrix}$ is an eigenvector of $\vec{A}$
[and *..count..* <u>most</u> exams last ten years have!]

then verification is easy: Multiply
$\vec{A}\begin{pmatrix}1\\2\\1\end{pmatrix}$ and see that you get $\lambda \begin{pmatrix}1\\2\\1\end{pmatrix}$,
some $\lambda \in \mathbb{R}$.

(b) If asked: $\lambda_2 = 4$ is an eigenvalue of $\vec{A}$.
Find a corresponding eigenvector $\vec{v}$:
Solve $(\vec{A} - 4\vec{I})\vec{v} = \vec{0}$. (You must get one (or more) degrees of freedom!)

(c) But how to find from scratch?

Note: $\vec{A}\vec{x} = \lambda \vec{x}$

$\Leftrightarrow \quad (\vec{A} - \lambda \vec{I})\vec{x} = \vec{0}.$

We want those λ for which there is a nonzero — i.e., non-unique! — solution.

That is: When $\underline{\det(\vec{A} - \lambda\vec{I}) = 0}$ (*)

(*) is called the <u>characteristic equation</u> of $\vec{A}$.

$p(\lambda) := \det(\vec{A} - \lambda\vec{I})$ is called the <u>characteristic polynomial</u>. It is an n^{th} order polynomial when $\vec{A}$ is $n \times n$. Leading coefficient: $(-1)^n$.

So, "method" for finding eigenvalues:

- Calculate $p(\lambda) = \det(\vec{A} - \lambda\vec{I})$
- Solve $p(\lambda) = 0$ for λ.

But ... if $n > 2$ ...? Or even > 4?

<u>If</u> we have found such a λ, its associated eigenvector is found by:

solving $(\vec{A} - \lambda\vec{I})\vec{x} = \vec{0}$ for $\vec{x}$, with this λ here.

2×2: We'll do the general case later.

Example: $\vec{A} = \begin{pmatrix} 2 & 3 \\ 3 & -6 \end{pmatrix}$.

there will be a formula for 2×2!

$$p(\lambda) = \det \begin{pmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{pmatrix} = -(2-\lambda)(6+\lambda) - 9$$

$$= [\text{you do the math}] = (3-\lambda)(-7-\lambda)$$

Eigenvectors:
the zeroes : 3 and -7 (as we have seen....)

Call them: $\mu = 3$ $\lambda = -7$

For $\mu = 3$: $\vec{A} - 3\vec{I} = \begin{pmatrix} -1 & 3 \\ 3 & -9 \end{pmatrix}$

Note: Proportional rows! Must be, because of order 2×2 and $\det = 0$.

$$(a \quad b)\begin{pmatrix} x \\ y \end{pmatrix} = 0 \quad \text{for } \begin{pmatrix} x \\ y \end{pmatrix} = t\begin{pmatrix} b \\ -a \end{pmatrix}$$

Eigenvectors corresponding to $\mu = 3$: $t\begin{pmatrix} 3 \\ 1 \end{pmatrix}$, any $t \neq 0$.

For $\lambda = -7$: $\vec{A} - (-7)\vec{I} = \begin{pmatrix} 9 & 3 \\ 3 & 1 \end{pmatrix}$

First row superfluous. Eigenvectors: $s\begin{pmatrix} 1 \\ -3 \end{pmatrix}$, any $s \neq 0$.

Note: Some row <u>must</u> be superfluous. This speeds up - or can be used to check calculations. Also, it it easy to <u>verify</u> an eigenvector $\vec{v}$: Just calculate $\vec{A}\vec{v}$

3×3 example:

An exam problem could be something not-too-unlike the following:

Let $\vec{A} = \begin{pmatrix} 1 & 3 & 1 \\ 3 & 1 & -1 \\ 1 & -1 & -3 \end{pmatrix}$

(a) $\vec{u} = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$ is an eigenvector of $\vec{A}$

Find the associated eigenvalue λ_1

(b) $\lambda_2 = 4$ is an eigenvalue of $\vec{A}$.

Find an associated eigenvector $\vec{v}$.

(c) Find a third eigenvector $\vec{w}$ such that $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly independent, or show that no such $\vec{w}$ exists.

(d) Decide the definiteness of $\vec{A}$.

[(d) is for next week!]

How to do these questions?

(a) Easy! $\vec{A}\begin{pmatrix}1\\-1\\-2\end{pmatrix}=\begin{pmatrix}-4\\4\\8\end{pmatrix}=\underbrace{-4}\ \vec{u}$

$\lambda_1=-4$

(b) $\lambda_2=4$: $(\vec{A}-4\vec{I})=\begin{pmatrix}-3&3&1\\3&-3&-1\\1&-1&-7\end{pmatrix}$ delete 2nd row

Gaussian elim: $\begin{pmatrix}0&0&-20\\1&-1&-7\end{pmatrix}$ $x_3=0,\ x_1=x_2,$

An eigenvector: $\vec{v}=\begin{pmatrix}1\\1\\0\end{pmatrix}$

(c) All right ... * sigh * ...

$$\begin{vmatrix}1-\lambda&3&1\\3&1-\lambda&-1\\1&-1&-3-\lambda\end{vmatrix}\overset{\text{Cofactor exp}}{=}[\dots]=-\lambda^3-\lambda^2+16\lambda+16$$

$$=-\lambda^2(\lambda+1)+16(\lambda+1)$$

so $\lambda_3=-1$. (More later.)

$\lambda_3=-1$: $\vec{A}+\vec{I}=\begin{pmatrix}2&3&1\\3&2&-1\\1&-1&-2\end{pmatrix}$ Can we delete?

$\sim\begin{pmatrix}0&5&5\\1&-1&-2\end{pmatrix}$ $x_3=t,\ x_2=-t,\ x_1=x_2+2x_3=t$

An eigenvector: $w=\begin{pmatrix}1\\-1\\1\end{pmatrix}$

Lin. indep? $\begin{vmatrix}1&1&1\\-1&1&-1\\-2&0&1\end{vmatrix}=\begin{vmatrix}0&1&1\\0&1&-1\\1&0&1\end{vmatrix}\neq 0.$

But fact: different eigenvalues $\Rightarrow$ lin. indep. eigenvectors!

Before we do the 2×2 case, two/three pieces of terminology:

<u>trace</u>: $\operatorname{tr}(\vec{A}) = a_{11} + \ldots + a_{nn}$.
$= $ sum of the main diag. elements.

Do <u>not</u> confuse "tr" with transpose.
(Some authors use "$\operatorname{sp}(\vec{A})$".)

<u>double root</u> of a polynomial $p(\lambda)$
A λ^* s.t. $q(\lambda) = \dfrac{p(\lambda)}{(\lambda^*-\lambda)^2}$ is
a polynomial, and $q(\lambda^*) \neq 0$.
... triple, quadruple, ...

<u>double eigenvalue</u>: double root of $|\vec{A} - \lambda \vec{I}|$.

<u>The general 2×2 case:</u> $\vec{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$,
$\operatorname{tr}\vec{A} = a+d$

$$|\vec{A} - \lambda\vec{I}| = (a-\lambda)(d-\lambda) - bc$$
$$= \lambda^2 - \lambda \operatorname{tr}\vec{A} + \det\vec{A}.$$

Eigenvalues: $\lambda = \dfrac{\operatorname{tr}\vec{A}}{2} \pm \sqrt{\left(\dfrac{\operatorname{tr}\vec{A}}{2}\right)^2 - \det\vec{A}}$

Q: When do we have two distinct, one double or no real eigenvalue(s) — and, their signs?

$$\lambda = \frac{\operatorname{tr}\vec{A}}{2} \pm \sqrt{\left(\frac{\operatorname{tr}\vec{A}}{2}\right)^2 - \det\vec{A}}$$

- If $\det\vec{A} < 0$ there are <u>two</u> eigenvalues <u>of opposite sign</u> $\lambda_2 > 0 > \lambda_1$. [has applications!

- If $\det\vec{A} = 0 \neq \operatorname{tr}\vec{A}$, we have two distinct eigenvalues 0 and $\operatorname{tr}\vec{A}$.

- If $0 < \det\vec{A} < \left(\frac{\operatorname{tr}\vec{A}}{2}\right)^2$ (i.e, if $\left(\frac{a-d}{2}\right)^2 + bc > 0$ — show that!) we have two distinct real eigenvalues, both of the same sign as $\operatorname{tr}\vec{A}$.

- If $\det\vec{A} = \left(\frac{\operatorname{tr}\vec{A}}{2}\right)^2$: double eigenvalue $\lambda = \frac{\operatorname{tr}\vec{A}}{2}$, (one or two lin. indep. eigenvectors)

- If $\det\vec{A} > \left(\frac{\operatorname{tr}\vec{A}}{2}\right)^2$: no real eigenvalue(s) (but two distinct complex ... two words on that).

Terminology: $z = x + y \cdot \sqrt{-1}$ has <u>real part</u> x, <u>imaginary part</u> y.

will be used. The real part of λ is

$$\begin{cases} \lambda & \text{if } \lambda \in \mathbb{R} \\ \frac{a+d}{2} & \text{if } \det\vec{A} > \left(\frac{\operatorname{tr}\vec{A}}{2}\right)^2 \end{cases}$$

case $n=2$

On $\left(\frac{\mathrm{tr}\,\vec{A}}{2}\right)^2 - \det \vec{A}$

$$= \frac{a^2+d^2}{4} + \underbrace{\frac{ad}{2} - ad}_{} + bc = \left(\frac{a-d}{2}\right)^2 + bc.$$

If b, c have same sign: real eigenvalue(s)

If opposite: only if a and d differ enough.

<u>If</u> real eigenvalue(s)

$$\vec{A} - \lambda\vec{I} = \begin{matrix} a - \frac{1}{2}\mathrm{tr}\vec{A} \mp \sqrt{\ldots} & b \\ c & d - \frac{1}{2}\mathrm{tr}\vec{A} \mp \sqrt{\ldots} \end{matrix}$$

$$= \begin{pmatrix} \frac{a-d}{2} \mp \sqrt{\left(\frac{a-d}{2}\right)^2 + bc} & b \\ c & \frac{d-a}{2} \mp \sqrt{\left(\frac{d-a}{2}\right)^2 + bc} \end{pmatrix}$$

(and some row can be deleted)

$$\lambda = \frac{tr\vec{A}}{2} \pm \sqrt{\frac{tr\vec{A}}{2} - \det\vec{A}}$$
$$= \frac{tr\vec{A}}{2} \pm \sqrt{\left(\frac{a-d}{2}\right)^2 + bc}$$

Examples! :)

$$\vec{A} = \begin{pmatrix} a & b \\ b & a \end{pmatrix}.$$

E-values $\lambda = a \pm \sqrt{\left(\frac{0}{2}\right)^2 + b \cdot b} = a \pm b.$

E-vectors: $\lambda_1 = a - b$ yields

$$\vec{A} - \lambda_1 \vec{I} = \begin{pmatrix} b & b \\ b & b \end{pmatrix}, \quad \underline{\underline{s\begin{pmatrix} 1 \\ -1 \end{pmatrix}}} \quad (s \neq 0)$$

$\lambda_2 = a + b$ yields

$$\vec{A} - \lambda_2 \vec{I} = \begin{pmatrix} -b & b \\ b & -b \end{pmatrix}, \quad \underline{\underline{t\begin{pmatrix} 1 \\ 1 \end{pmatrix}}} \quad (t \neq 0)$$

$$\vec{A} = \begin{pmatrix} 123456 & -54 \\ 32 & 123456 \end{pmatrix}$$

$a = d$ and $bc < 0$, $\det > \left(\frac{tr}{2}\right)^2$,

no (real) eigenvalue.

(But... real part $= \frac{tr\vec{A}}{2} = 123456\ldots$)

$n \times n$: General facts.

- The characteristic polynomial $p(\lambda) = |\vec{A} - \lambda \vec{I}|$ is of form

two words on "$|\lambda\vec{I} - \vec{A}|$"

$$(-1)^n \lambda^n - \lambda^{n-1} \cdot \operatorname{tr} \vec{A} + K_{n-2} \lambda^{n-2} + \ldots + \underbrace{\det \vec{A}}_{= p(0)}$$

- Any n^{th} order polynomial with leading coeff $= (-1)^n$, can be written

$$\boxed{(\lambda_1 - \lambda) \cdot \ldots \cdot (\lambda_k - \lambda) \cdot \prod_{i=1}^{\ell} (\lambda^2 + b_i \lambda + c_i)}$$

product

with $c_i > (b_i/2)^2$ (no real zeroes)

$k \geq 0$ terms (0 possible)

$\ell \geq 0$ terms, 0 possible

$k + 2\ell = n$

Then $\lambda_1, \ldots, \lambda_k$ are the real roots.

A number that repeats precisely m times in this list, is a root of <u>multiplicity m</u>

For eigenvalues: often "algebraic multiplicity m" as such an eigenvalue can have $\gamma \leq m$ lin. indep. eigenvectors (γ = "geometric multiplicity")

Do not bother, except: <u>beware</u> that a double eigenvalue can have $\gamma = 1$ or $\gamma = 2$.

Facts on sums/products of eigenvalues

- If $p(\lambda) = (\lambda_1 - \lambda) \cdot \ldots \cdot (\lambda_n - \lambda)$ ⊛

then $\lambda_1 + \ldots + \lambda_n = \operatorname{tr} \vec{A}$

$\lambda_1 \cdot \ldots \cdot \lambda_n = \det \vec{A}$

remember multiplicity!
E.g. if $p(\lambda) = (3-\lambda)^2$
then $\lambda_1 + \lambda_2 = 3+3 = 6$
$\lambda_1 \cdot \lambda_2 = 3 \cdot 3 = 9$

Can be used to find eigenvalues!

- If you accept to work with complex numbers (→ not required at the exam) then ⊛ is true.

If you do not want complex numbers, then you could hack the result to work, e.g.:

if $p(\lambda) = (5-\lambda)(\lambda^2 + 4\lambda + 13)$

$\lambda_1 = 5$

$\lambda_{2,3} = -2 \pm 3\sqrt{-1}$

real part $= -2$. Use that in the sum

$\operatorname{tr} \vec{A} = 5 + (-2) + (-2) = 1$

(For $\det \vec{A}$: Use $\lambda_2 \cdot \lambda_3 = 13$ (the "c" in $\lambda^2 + b\lambda + c$!)

$\det \vec{A} = 5 \cdot 13 = 65$.)

But the non-real case is not so useful if you only want to find real eigenvalues!

So, back to the "exam-alike" problem:

$$\vec{A} = \begin{pmatrix} 1 & 3 & 1 \\ 3 & 1 & -1 \\ 1 & -1 & -3 \end{pmatrix}, \quad p(\lambda) = -\lambda^3 - \lambda^2 + 16\lambda + 16.$$

- If you did not spot that $p(-1) = 0$
  [it does not hurt much to *try* a small integer?]
  you can find it as follows:
  - It is *given* in (b) that $p(4) = 0$.
  - Divide: $\left(-\lambda^3 - \lambda^2 + 16\lambda + 16\right) : (\lambda - 4)$
  - Solve the quadratic.
- If you did also solve part (a) – the easy – you know that $p(-4) = 0$. So
  $(\lambda - 4)(\lambda + 4) = \lambda^2 - 16$ is a factor.

$$\begin{array}{l} \left(-\lambda^3 - \lambda^2 + 16\lambda + 16\right) : \lambda^2 - 16 = -\lambda - 1 \\ \underline{-\lambda^3 \qquad + 16\lambda} \\ \qquad -\lambda^2 \qquad + 16 \end{array}$$

- But you could also use
$$\lambda_1 + \lambda_2 + \lambda_3 = \operatorname{tr} \vec{A} = -1,$$
$$\lambda_1 \lambda_2 \lambda_3 = \det \vec{A} = 16.$$

If you only know $\lambda_1 = 4$:
$$\lambda_2 + \lambda_3 = -5, \quad \lambda_2 \cdot \lambda_3 = 4$$
yield -1 and -4. If you know $\lambda_2 = -4$, it is even easier!