Differential eq's (ordinary, not partial)

ordinary derivatives only — involving partial derivatives

Math 2: first order.

→ separable

→ linear

Math 3:

→ one non-separable nonlinear first-order type (Bernoulli) if time permits

→ second-order: not the "2nd order generalization of separable" (autonomous...)

→ second-order linear

→ linear first-order systems $\dot{\vec{x}} = \vec{A}\vec{x} + \vec{b}$

Recall terminology:

→ particular solution

→ general solution

Recall 1^{st} order linear:

$$\dot{x} + a(t)\, x = f(t) \qquad \circledast$$

→ Formula

→ Can: $\frac{d}{dt}\left(x e^{A(t)}\right) = \left(\dot{x} + \dot{A}(t)\, x\right) e^{A(t)}$

Let $\dot{A}(t) = a(t)$; then this equals

$f(t)\, e^{A(t)}$. Integrate:

$$x\, e^{A(t)} = C + \underbrace{\int f(t) e^{A(t)}\, dt}_{\substack{\text{any antiderivative,} \\ \text{when we write the "C"}}}$$

A could be any antiderivative of a.

→ Note:

The general solution of $\circledast$

= the general solution of $\dot{x} + a(t)x = 0$

(the "corresponding homogeneous equation")

+

any particular solution of $\circledast$.

Ex: $\dot{x} + 2x = 3$; $\dot{x} + 2x = 0$ has general solution Ce^{-2t}

$x = \frac{3}{2}$ is a particular solution

So: $x = \underline{\frac{3}{2} + Ce^{-2t}}$

Compare with algebraic eq. $\vec{M}\vec{x} = \vec{b}$, $\circledast\circledast$

$\vec{x}$ = general sol'n of $\vec{M}\vec{x} = \vec{0}$ + some solution of $\circledast\circledast$

2^{nd} order:

$$\ddot{x} = F(t, x, \dot{x}) \qquad \Big| \qquad (\,G(t, x, \dot{x}, \ddot{x}) = 0 \ldots)$$

Example:

$\ddot{x} = 2$ Gen. sol.: $\ddot{x} = t^2 + Ct + D$.

* TWO constants!

→ No "separable" diff. eq;

→ but the <u>autonomous</u> ones, $\ddot{x} = F(x, \dot{x})$

have a somewhat ~~related~~ method

Sketch:

$y = \dot{x}$, $\quad \dfrac{dy}{dt}\left(\dfrac{dt}{dx}\,\underbrace{\dfrac{dx}{dt}}_{y}\right) = F(x, y)$

$$y\,\frac{dy}{dx} = F(x, y)$$

solve for y as function of x (disregard that x was originally the unknown!),

get $y = H(x)$. $y = \dot{x}$, so

$\dot{x} = H(x)$ to solve for x.

→ Focus of this course: <u>linear</u>:

$$\ddot{x} + a(t)\dot{x} + b(t)x = f(t). \qquad (*)$$

→ will soon assume a, b constant, but not f!

Again:

Homogeneous eq.

$$\ddot{x} + a\dot{x} + bx = 0 \qquad (H)$$

(*) has general sol. =

gen. sol of (H)

+ <u>some</u> sol.** of (*).

So we proceed to solve

(H) $\ddot{x} + a\dot{x} + bx = 0$ ("$\neq f$" to be covered later)

Fact: If we have two <u>non-proportional</u> particular solutions u_1 and u_2 of (H), then the general solution is

$$C_1 u_1(t) + C_2 u_2(t).$$

Example: $\ddot{x} = \delta^2 x$

Hint: $e^{\delta t}$

$\delta^2 \cancel{e^{\delta t}} = \delta^2 \cancel{e^{\delta t}}$

$C_1 e^{\delta t} + C_2 e^{-\delta t}$

→ $C_1 u_1 + C_2 u_2 + ct$ holds whether const. coeffs or not.

→ One 2nd order homogeneous with non-constant coeff's:

$$t^{p+2}\ddot{x} + \alpha t^{p+1}\dot{x} + \beta t^{p} x = 0$$

t^q: $$\left[q(q-1) \cancel{t^{p+q}} + \alpha q + \beta \right] t^{p+q} = 0$$

$$q^2 + (\alpha - 1) q + \beta = 0$$

$$q = \frac{1}{2}\left[1 - \alpha \pm \sqrt{(1-\alpha)^2 - 4\beta} \right]$$

Two solutions q_1, q_2 if $(1-\alpha)^2 > 4\beta$.

→ ~~Do~~ We skip the other ~~two~~ cases

Constant coeff's. "= f" later.

$\ddot{x} + a\dot{x} + b = 0$

Try e^{rt}.

$r^2 e^{rt} + a r e^{rt} + b e^{rt} = 0$

$r^2 + ar + b = 0$

$r = -\frac{a}{2} \pm \sqrt{\left(\frac{a}{2}\right)^2 - b}$

- So if $\left(\frac{a}{2}\right)^2 - b > 0$ then OK:

$\underline{C_1 e^{r_1 t} + C_2 e^{r_2 t}}$.

- if $\left(\frac{a}{2}\right)^2 = b$? $(At + B)e^{rt}$ one r

Insert & see it works!

Why? $(C_1 + C_2)e^{r_1 t} + C_2(-e^{r_1 t} + e^{r_2 t})$

Put $C_2 = \frac{A}{r_2 - r_1}$ $\quad r_2 - r_1 = \underbrace{2\sqrt{}}_{h}$

$\frac{e^{(r_1 + h)t} - e^{r_1 t}}{h} \to \underline{t}$

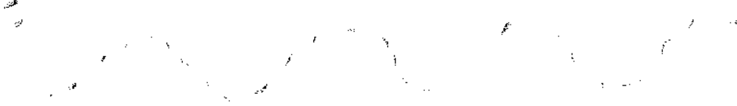
Q: What if $\left(\frac{a}{2}\right)^2 < b$?

Consider $\ddot{x} = -bx$, $b > 0$.

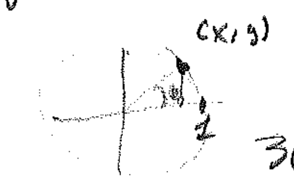
→ concave when $x > 0$

→ convex when $x < 0$

In fact:



Trigonometrics:



θ = ratio arclength/radius

$360° =$ angle of 2π.

Let radius = 1, so (x, y) satisfies $x^2 + y^2 = 1$

Define the functions "cos" and "sin" by

$\cos\theta = x$ | x/radius

$\sin\theta = y$ | y/radius.

~~No~~ Trigs has some odd notation:

$\cos 2\theta$ for $\cos(2\theta)$

$\cos^2\theta$ for $(\cos(\theta))^2$

etc.

Trigonometry for differential eq.'s

Want: To solve diff. eq's of the form

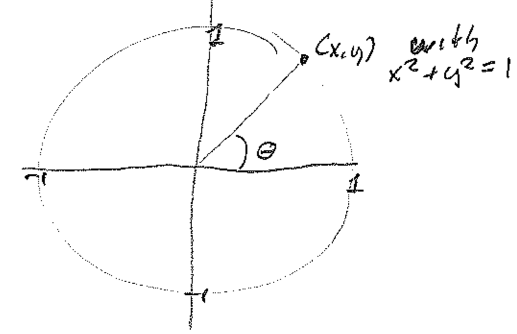
$$\ddot{x} + a\dot{x} + bx = f(t).$$

Special case: $\ddot{x} + bx = 0$

Has solutions $x = e^{\pm\sqrt{|b|}\,t}$ if $b < 0$.

Need other functions if $b > 0$.

Sin & cos: "Unit circle definitions"



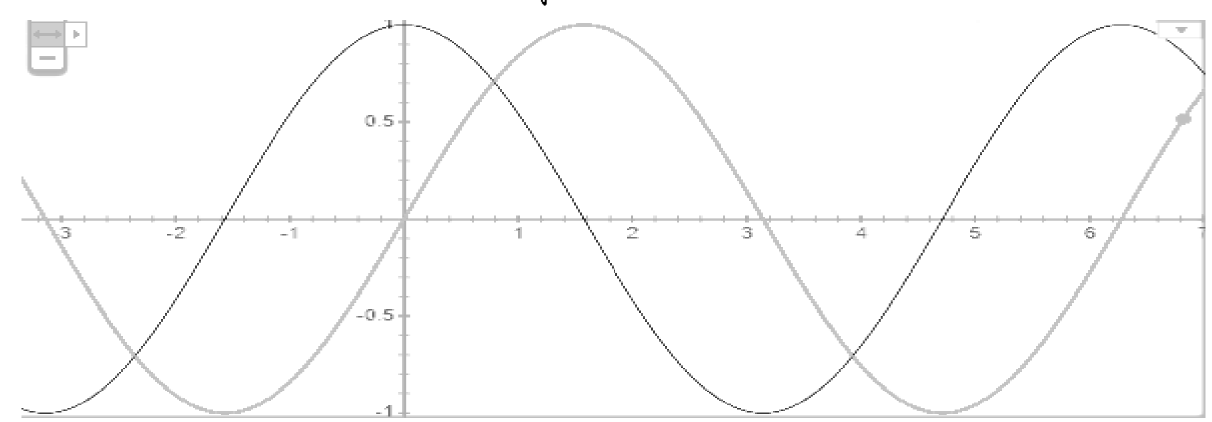
θ in radians, i.e.

θ = arc length

Def: $\cos\theta$ = the x coordinate $\in [-1,1]$

$\sin\theta$ = the y coordinate. $\in [-1,1]$

Plots: dark = cos, lighter but thicker = sin



$$\cos\theta = \sin\left(\theta + \frac{\pi}{2}\right) \quad ; \quad \sin\theta = -\cos\left(\theta + \frac{\pi}{2}\right)$$

Some facts

Periodicity $\quad \sin(\theta \pm 2\pi) = \sin\theta \qquad \cos(\theta \pm 2\pi) = \cos\theta$

Translates of each other (prev page) $\quad \sin\theta = \cos(\theta - \frac{\pi}{2}) \qquad \cos\theta = \sin(\theta + \frac{\pi}{2})$

Concavity/convexity? $\quad$ Inflect at the zeroes

odd/even? $\quad$ Odd: $\sin(-\theta) = -\sin\theta \qquad$ even: $\cos(-\theta) = \cos\theta$

Lin. comb... $\quad$ Any $A\cos\theta + B\sin\theta$ can be written

$$= C\cos(\omega + \theta) \qquad = D\sin(\varphi + \theta)$$

Derivatives? $\quad \sin' = \cos,\ \sin'' = -\sin \qquad \cos' = -\sin,\ \cos'' = -\cos$

"Inverses": $\quad \sin^{-1}$ = the inverse of $\sin\theta$; $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ $\qquad \cos^{-1}$ = the inverse of $\cos\theta$; $\theta \in [0, \pi]$

(other) algebraic formulae: $\quad$ a lot! Only a few here.

$$(\sin\theta)^2 + (\cos\theta)^2 = 1.$$

Typical notation: $\quad \cos^2\theta$ for $(\cos\theta)^2$ etc

$\cos 3t$ for $\cos(3t)$ etc

Example:
$$\int \frac{\cos\theta}{\sin\theta}\, d\theta = \int \frac{du}{u} = C + \ln|u| = \underline{C + \ln|\sin\theta|}$$

$u = \sin\theta$
$du = \cos\theta\, d\theta$

Also:
$$\int \frac{1}{\sqrt{1-x^2}}\, dx = \int \frac{\cos\theta\, d\theta}{\sqrt{\cos^2\theta}} = \int \pm 1\, d\theta = C \pm \sin^{-1} x \ \Big|\ \pm ?$$

$x = \sin\theta$; $1 - x^2 = 1 - \sin^2 = \cos^2$
$dx = \cos\theta\, d\theta$

$$= \underline{C + \sin^{-1} x}$$

after testing.

sin & cos: "series definition"

$$\exp\theta = 1 + \theta + \frac{\theta^2}{2} + \frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \dots$$

Turns out: $\exp(\theta) = e^{\theta}$, $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$.

Define:

$$\cos\theta = 1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots$$

(Every other term; alternating signs.)

If you differentiate term-by-term
[which is OK here, but not obviously ok!]
you get $\exp' = \exp$, $\sin' = \cos$, $\cos' = -\sin$.

(If you accept complex numbers: Let $i^2 = -1$.
Then $\cos(i\theta) = \Sigma$ even-order terms, all with $+$
$\sin(i\theta) = i \cdot \Sigma$ odd-order terms with $+$
$\exp(i\theta) = \cos\theta + i\sin\theta$
and famously, Euler's formula

$$e^{i\pi} + 1 = 0$$

"the most remarkable formula in mathematics", quoting Feynman)