Homogeneous 2^{nd} order diff. eq's with <u>constant</u> coefficients

$$\ddot{x} + a\dot{x} + bx = 0$$

a, b constant.

* <u>Need</u>: two non-proportional particular solutions $u_1(t)$, $u_2(t)$.

Then: $x(t) = C_1 u_1(t) + C_2 u_2(t)$

* Initial value problems: Need <u>two</u> data to determine C_1, C_2. Ex.: $x(t_0)$, $\dot{x}(t_0)$.

Then
$$\begin{pmatrix} u_1(t_0) & u_2(t_0) \\ \dot{u}_1(t_0) & \dot{u}_2(t_0) \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} x(t_0) \\ \dot{x}(t_0) \end{pmatrix}$$

determines C_1, C_2.

Sometimes, C_1 and C_2 are determined not by initial cond's, but, say, boundary cond's: $x(t_0)$ and $x(t_1)$ given.

<u>Cookbook</u> for

$\ddot{x} + a\dot{x} + bx = 0$ (a, b constant)

* Form the <u>characteristic polynomial</u>

$r^2 + ar + b$.

familiar name - no coincidence.

* Zeroes:

$r = -\frac{a}{2} \pm \sqrt{\left(\frac{a}{2}\right)^2 - b}$

Three cases:

(2) Two distinct real roots r_1, r_2.

Then $u_1(t) = e^{r_1 t}$, $u_2(t) = e^{r_2 t}$ work.

General sol'n: $C_1 e^{r_1 t} + C_2 e^{r_2 t}$.

(1) Double root r: e^{rt} and te^{rt}.

$e^{rt} \cdot (A + Bt)$

(0) No real root: case $b - \left(\frac{a}{2}\right)^2 > 0$

Real part $\alpha = -\frac{a}{2}$. Put $\beta := \sqrt{b - (a/2)^2}$

("imaginary part")

Solution: $e^{\alpha t} \cdot \left(A\cos(\beta t) + B\sin(\beta t)\right)$.

Can also write: $Ce^{\alpha t}\cos(\omega + \beta t)$

arbitrary constants

Why ②? Try e^{rt}. Insert. Get

$$\cancel{e^{rt}}(r^2+ar+b)=0. \quad \text{Fit } r.$$

Why ①? Again, we can insert. e^{rt} as above.

$u=te^{rt}$? $\quad \dot{u}=(1+rt)e^{rt} \quad \ddot{u}=\big(r+r(1+rt)\big)e^{rt}$

Get

$$\cancel{te^{rt}(r^2+ar+b)}+\underbrace{(2r+a)}_{\text{and } r=-a/2.}e^{rt}$$

But also note: With two distinct roots,

we have $C_1e^{-\frac{a}{2}t}\left(e^{ht}-e^{-ht}\right)+(C_1+C_2)e^{-(\frac{a}{2}+h)t}$

where $h=\sqrt{(\frac{a}{2})^2-b}$.

When $h\approx 0$, $\quad e^{ht}-e^{-ht}\approx 2ht$.

$$\underbrace{2C_1h}_{B}\,te^{-\frac{a}{2}t}+\underbrace{(C_1+C_2)}_{A}e^{-rt}.$$

Why ③? Again, insert and check.

But the deeper explanation: oscillations and growth are in a sense the same phenomenon!

Ex:

$$2\ddot{x} - 8\dot{x} + 8x = 0$$

i.e. $$\ddot{x} - 4\dot{x} + 4x = 0$$

Solve $$\underbrace{r^2 - 4r + 4}_{(r-2)^2} = 0$$

Double root $r = 2$.

Solution: $(A + Bt)e^{2t}$ Cst

"Initial conditions":

→ 1^{st} order: ~~Need~~ $x(t_0) = x_0$ determines the constant

→ 2^{nd} order: Need two data.

Could be

$$\begin{cases} x(t_0) = x_0 \\ x(t_1) = x_1 \end{cases}$$

or

$$\begin{cases} x(t_0) = x_0 \\ \dot{x}(t_1) = v \end{cases}$$ typically with $t_0 = t_1$.

Ex: (*) ~~…~~ with $x(0) = \dot{x}(0) = 1$

$A e^{2\cdot 0} = 1$ $\quad \underline{A = 1}$

$1 = (1 + Bt)2e^{2t} + Be^{2t}\big|_{t=0}\ 2 + B$, $\underline{B = -1}$

Note for initial cond's: optional, but cond/save time

Suppose we are given $x(t_0) = x_0$, $\dot{x}(t_0) = v$

Case ⓪

$$x(t) = e^{\alpha(t-t_0)}\left(x_0 \cos(\beta\cdot(t-t_0)) + B\cdot \sin(\beta\cdot(t-t_0))\right)$$

has $x(t_0) = x_0$. To fit B:

$$\dot{x} = \alpha\, x(t) + e^{\alpha(t-t_0)}\left(-x_0 \sin\beta(t-t_0) + B\cos\beta(t-t_0)\right)$$

$$v = \alpha\, x_0 + 1\cdot(0 + B)$$

$$B = v - \alpha x_0$$

It simplifies to have a decomposition with even and odd function; if $\alpha = 0$, then the constants are the initial data!

Case ① $x(t) = e^{r\cdot(t-t_0)}(x_0 + Bt)$ has $x(t_0) = x_0$.

To fit B: $\dot{x}(t_0) = r x_0 + B$; $B = v - r x_0$
$= v + \frac{a}{2}x_0$

Case ② Here it helps to write $C_1 e^{r_1 t} + C_2 e^{r_2 t}$

as $\underbrace{A}_{x_0} \underbrace{\frac{e^{r_1(t-t_0)} + e^{r_2(t-t_0)}}{2}}_{=1 \text{ at } t_0} + B \underbrace{\frac{e^{r_1(t-t_0)} - e^{r_2(t-t_0)}}{2}}_{=0 \text{ at } t_0}$ because this $= 0$ at $t_0 = 0$.

$$\dot{x}(t_0) = x_0 \underbrace{\frac{r_1 + r_2}{2}}_{= -\frac{a}{2}} + B \underbrace{\frac{r_1 - r_2}{2}}_{= \sqrt{(\frac{a}{2})^2 - b}\ > 0}, \text{ so}$$

$$B = \left(v + \frac{a}{2}x_0\right)\cdot \frac{1}{\sqrt{(\frac{a}{2})^2 - b}}$$

Particular solutions u^* of

$$\ddot{x} + a\dot{x} + bx = f(t), \quad b \neq 0$$

* If $f(t)$ is a polynomial, try u^* = polynomial, same degree.

* If $f(t) = e^{qt}$ try e^{qt}
* $f(t) = P_n(t)e^{qt}$ try $Q_n(t)e^{qt}$
* If $f(t) = \sin qt$ or $\cos qt$

try $u^* = \underbrace{K\cos qt + L\sin qt}_{\text{You need BOTH !}}$

* If $f(t)$ = sum of these, try $u^*(t)$ = sum of these.

Ex: $\ddot{x} - 4\dot{x} + 4x = \sin 3t$

$u^* = K\cos 3t + L\sin 3t$

$\dot{u}^* = -3K\sin 3t + 3L\cos 3t$

$\ddot{u}^* = -9u$

$K = \frac{12}{169}$

$L = -\frac{5}{169}$

LHS $= \cancel{9K}\cos 3t \cdot (-9K + 3L\cdot(-4) + 4K)$

$+ \sin 3t\,(-9L - 3K\cdot(-4) + 4L)$

should match $\sin 3t$ $\cdot 5K + 12L = 0$

$12K - 5L = 1$

$\cdot 5$

$\cdot 12$

Q: What if f solves the homogeneous?

I $\quad f = e^{rt}$ $\quad$ r complex

II or possibly, for $r = -a/2$, $(\frac{a}{2})^2 = b$: $t e^{rt}$

I: Try $u^* = K t e^{rt}$

$$\dot{u}^* = K[t r e^{rt} + e^{rt}]$$
$$\ddot{u}^* = K[t r^2 + 2r] e^{rt}$$

$$\ddot{u}^* + a\dot{u}^* + bu =$$
$$K e^{rt} \cdot [t \cdot [\cancel{r^2 + ar + b}] + 2r + a]$$

so $K = \frac{1}{2r+a}$ for $r \neq -a/2$.

For $r = -a/2$: $f = e^{rt}$ yields $u^* = K t^2 e^{rt}$.

.... a bit of work! Even more when $r = -a/2$, $f = t e^{-at/2}$.

~~t^2~~ third order ...