Dynamic optimization, continuous time

No longer "find the best point on a graph";

this is "find the best function".

Typical problem to consider:

consumption – saving trade-off.

State $= x(t)$. Consuming increases direct utility now, but reduces $\dot{x}$ and thus the potential for future consumption.

Tools on curriculum:

1696–1750s tool

* The Euler equation of calculus of variations
  - → you control $\dot{x}$,
  - → a F.O.C. leads to a 2^{nd} order diff. eq.

* Pontryagin's maximum principle (1956 ff. tool)
  - → $\dot{x} = g(t, x, u)$, you control u, could be restricted (can have boundary optimum)
  - → weight p on future: direct utility $+ pg$ ...
  - → leads to a system $\begin{pmatrix} \dot{x} \\ \dot{p} \end{pmatrix}$ = transformation of (t, x, p)

Recall: 2^{nd} order diff. eq. in $\mathbb{R}^1$

$\updownarrow$

1^{st} order system in $\mathbb{R}^2$

  - → More general.

Calculus of variations & the Euler equation

Problem: $\max/\min \displaystyle\int_{t_0}^{t_1} F(t, x(t), \dot{x}(t))\, dt$, $x(t_0) = x_0$, $x(t_1) = x_1$

where t_0, t_1, x_0, x_1 are given, and we maximize over all C^1 functions x starting at $x(t_0) = x_0$, ending at $x(t_1) = x_1$.

(or maybe just piecewise)

Notes on notation:

→ Book uses F for running utility in this chapter, f in the next.

→ F "must" depend on $\dot{x}$, otherwise there is no dynamic trade-off.
Use $\frac{\partial F}{\partial \dot{x}}$ for partial derivative wrt 3^{rd} variable
Alternatively: $\frac{\partial F}{\partial u}$, $(u = \dot{x})$ – or F_3'.

There are tools for cases "$x(t_1)$ free" or "$\geq x_1$", but we cover those under the maximum principle.

The <u>Euler equation</u> for the problem:

$$\frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = 0$$

<u>total</u> derivative, note that $x = x(t)$,
$\dot{x} = \dot{x}(t)$.

(The Euler eq. is 2^{nd} order if $F''_{32} \neq 0$:

$$F''_{33}\ddot{x} + F''_{32}\dot{x} + F''_{31} - F'_2 = 0$$

(harder to remember!)).

Fact: Let $F \in C^2$.

Necessary cond's: The Euler eq & $x^*(t_0) = x_0$
$x^*(t_1) = x_1$.

Sufficient: The necessary & in addition
$(x, \dot{x}) \to F$ concave, each $t \in (t_0, t_1)$ for max
$(x, \dot{x}) \to F$ convex —— , —— min.

Some special cases:

- F has no x-dependence: $\frac{d}{dt}\frac{\partial F}{\partial \dot{x}} = 0 \Rightarrow \frac{\partial F}{\partial \dot{x}} = C$
- F has no t-dependence: $F - \dot{x}\frac{\partial F}{\partial \dot{x}} = C$ (will not show this)

First order! "First integral".

Example (bode):

$$\min \int_0^T \underbrace{\left(x^2 + c\dot{x}^2\right)}_{\text{convex in } (x,\dot{x})} dt, \qquad \begin{array}{l} c>0 \text{ constant} \\ x(0) = x_0 \text{ given} \\ x(T) = 0. \end{array}$$

notation: means "$(x(t))^2 + c(\dot{x}(t))^2$".

* Euler eq.:

$$0 = \underbrace{2x}_{\frac{\partial F}{\partial x}} - \frac{d}{dt}\underbrace{(2c\dot{x})}_{\frac{\partial F}{\partial \dot{x}}} = 2x - 2c\ddot{x}$$

yields $\ddot{x} - \frac{1}{c}x = 0$. $r^2 - \frac{1}{c} = 0$. Put $\rho = c^{-1/2}$

General sol'n $Ae^{\rho t} + Be^{-\rho t}$.

Fit constants: $A + B = x_0$

$Ae^{\rho T} + Be^{-\rho T} = 0$ [...]

yields $x^*(t) = x_0 \dfrac{e^{\rho(T-t)} - e^{-\rho(T-t)}}{e^{\rho T} - e^{-\rho T}}$

(* "Alt.:" By first integral: $C = F - \dot{x}\frac{\partial F}{\partial \dot{x}}$

$= x^2 + c\dot{x}^2 - \dot{x}\cdot 2c\dot{x}$

$= x^2 - c\dot{x}^2$

$\dot{x} = c^{-1/2}\sqrt{x^2 - C}$ not easy!

Check Wolfram Alpha for the solution, try to reverse-engineer by differentiating and integrate with the steps done backwards; still not easy!)

Example:

The arc length of a curve: $\int \sqrt{(dt)^2 + (dx)^2}$ (by Pythagoras) $= \int \sqrt{1 + \left(\frac{dx}{dt}\right)^2}\, dt$.

min/max $\displaystyle\int_0^T \sqrt{1 + \dot{x}^2}\, dt$, $\quad x(0) = x_0$, $x(T) = x_T$

Euler Eq: $\quad 0 = \overset{\partial F/\partial x}{0} - \frac{d}{dt} \frac{\partial}{\partial \dot{x}} \sqrt{1 + \dot{x}^2}$

So $\quad \frac{\partial}{\partial \dot{x}} \sqrt{1 + \dot{x}^2} = C$

$$\sqrt{1 + \dot{x}^2} = C\dot{x} + D$$

$\dot{x}$ depends only on constants $\rightsquigarrow$ $\dot{x}$ is a constant.

x = straight line. Surprised?

Application: Ramsey's consumption/savings problem

$$\underset{\text{output}}{Y} = f(\underset{\text{capital}}{K}) \qquad (f' > 0 \geq f'')$$

$$Y(t) = \underset{\text{consumption}}{c(t)} + \underset{\text{investment}}{\dot{k}(t)} \qquad c = f(k) - \dot{k}$$

• utility $U(c)$ from consumption, $U' > 0 > U''$. Problem:

$$\max \int_0^T \underbrace{U(f(k) - \dot{k})}_{\text{concave in } (k,\dot{k})} e^{-rt}\, dt \quad \text{(why?)} \qquad \text{s.t.} \quad \left.\begin{array}{l} k(0) = k_0 \\ k(T) = k_T \end{array}\right\} \text{given.}$$

discounted, $r \geq 0$

State $= k$.

Euler Eq.:

$$0 = \overbrace{U'(c) f'(k) e^{-rt}}^{\frac{\partial F}{\partial k}} - \frac{d}{dt}\left(U'(c) \cdot (-1) e^{-rt}\right)$$

$$= U'(c) f'(k) e^{-rt} + U''(c)\dot{c} e^{-rt} - r U'(c) e^{-rt}$$

$$0 = U'(c) \cdot (f'(k) - r) + \dot{c}\, U''(c).$$

often written:

$$\frac{\dot{c}}{c} = (r - f'(k)) \Big/ \underbrace{El_c U'(c)}_{\text{elast. of mg utility}, <0.}$$

Consumption <u>increases</u> as long as $f'(k) - r > 0$,

i.e. mg. prod. of capital > discount rate.

Can insert $\dot{c} = f'(k)\dot{k} - \ddot{k}$ .....

If time: <u>Why</u> the Euler eq.?

Consider a path $x = x(t)$. Modify it: $x + \underset{\uparrow\,\text{function}}{\overset{\text{constant}\,\downarrow}{\alpha\,\mu}}$.

(the "variation" in "calculus of variations")

Get:

$$\int_{t_0}^{t_1} F(t,\, x+\alpha\mu,\, \dot{x}+\alpha\dot{\mu})\, dt$$

If $x = x^*$ is optimal, the "best" variation is 0.

So take $\frac{d}{d\alpha}$ and insert $\alpha = 0$; that must yield 0.

$$\int_{t_0}^{t_1} \left(\frac{\partial F}{\partial x}\mu + \frac{\partial F}{\partial \dot{x}}\dot{\mu} \right) dt = 0 \quad \text{when } x = x^*.$$

Trick: integrate by parts to "turn $\dot{\mu}$ into μ":

$$\int_{t_0}^{t} \frac{\partial F}{\partial \dot{x}}\dot{\mu}\, dt = \Big|_{t_0}^{t_1} \frac{\partial F}{\partial \dot{x}}\mu - \int_{t_0}^{t_1} \mu \frac{d}{dt}\frac{\partial F}{\partial \dot{x}}\, dt$$

Since $x^* + \alpha\mu = \begin{cases} x_0 & \text{at } t_0 \\ x_1 & \text{at } t_1, \end{cases}$
$\mu(t_0) = \mu(t_1) = 0$.

So

$$\int_{t_0}^{t_1} \mu(t) \left(\frac{\partial F}{\partial x} - \frac{d}{dt}\frac{\partial F}{\partial \dot{x}} \right) dt = 0 \quad \text{when } x = x^*,$$

to hold for <u>all</u> μ with $\mu(t_0) = \mu(t_1) = 0$.

But: Make sure that μ has same sign as $\frac{\partial F}{\partial x} - \frac{d}{dt}\frac{\partial F}{\partial \dot{x}}$

Then the latter must be 0 to get 0.