Continuous-time optimization:

Optimal control theory : Pontryagin's maximum principle

Quantities involved:

$u = u(t) \in U$ (U = fixed set) : our control, affecting the state variable x

$x = x(t)$: state variable, obeying a diff. eq.

$$\dot{x}(t) = g(t, x(t), u(t))$$

starting at $x(t_0) = x_0$ given

Terminal conditions: Let $t_1 > t_0$ fixed (for now); shall consider three types of terminal cond's.

(a) $x(t_1) = x_1$ (b) $x(t_1) \geq x_1$ (c) $x(t_1)$ free.

Objective / criterion:

$$\int_{t_0}^{t_1} f(t, x(t), u(t))\, dt \quad , \quad \text{to be maximized}$$

(otherwise, rewrite)

over all admissible pairs (x, u): That is, satisfying $u(t) \in U$, each t

$$\dot{x} = g(t, x, u) \quad , \quad x(t_0) = x_0$$

& the terminal condition.

Optimal pair (x^*, u^*): an admissible pair that maximizes the objective.

Idea:

- Attach weight 1 to running utility $f(t,x,u)$
  and p to future state: $\dot{x}$
- ... so that u^* maximizes $f(t,x,u) + p\,g(t,x,u)$
- Turns out: the "correct" p satisfies a diff. eq, involving x
  We get a diff. eq. system for $\begin{pmatrix}\dot{x}\\ \dot{p}\end{pmatrix} = \ldots$

How to:

Define the Hamiltonian H: (compare to Lagrangian!)

$$H(t,x,u,p) = f(t,x,u) + p\,g(t,x,u)$$

Conditions: (x^*, u^*) is admissible

the "maximum principle"

u^* maximizes H (over $u \in U$)

$\dot{p} = -\frac{\partial H}{\partial x}(t, x^*(t), u^*(t), p(t))$ with transversality cond's

(a) if terminal condition $x(t_1) = x_1$: no condition on $p(t_1)$

(c) if $x(t_1)$ free: $p(t_1) = 0$

(b) if terminal condition $x(t_1) \geq x_1$:
$p(t_1) \geq 0$, and $= 0$ if $x^*(t_1) > x_1$.

The maximum principle is necessary in case (c), and "usually" necessary in (a), (b); there is a "constraint qualification" quirk in case no weight can be attached to f (control must be reserved to fulfill terminal cond's)

$\Rightarrow$ not exam relevant.

- Sufficient conditions? Later! Examples first.

Example 1

$$\max_{u(t)\in[0,1]} \int_0^T e^{-\delta t} u(t)\, dt, \qquad \delta>0 \qquad \dot{x} = -u, \qquad x(0)=x_0>0, \quad x(T)\geq 0.$$

("Obvious" solution since $\delta>0$: $u=1$ until $t=T$ or $x(t)=0$.)

Question: Find the admissible pair(s) satisfying the max. principle.

$H(t,x,u,p) = e^{-\delta t}u - pu$

u^* maximizes H: $\qquad u^* = \begin{cases} 1 & \text{if } p<e^{-\delta t} \\ 0 & \text{if } p>e^{-\delta t} \\ ? & \text{if } p(t)=e^{-\delta t} \end{cases}$

Note: if $p(t)=e^{-\delta t}$, we don't (yet?) have information about u^*.

$\dot{p} = -\frac{\partial H}{\partial x} = 0$, so $p\equiv\bar{p}$ constant,

with: $0 = p(T) = \bar{p}$ if $x^*(T)>0$

$0\leq p(T) = \bar{p}$ always.

Note: should $\bar{p} = e^{-\delta t}$, some t, then it is only one – and it does not matter for x nor the objective what u we choose at a single point in time!

Try $u^*\equiv 1$. OK if admissible, ie if $x_0 - T\geq 0$,

If $x_0>T$ then $x(T)>0$ and $\bar{p}=0$.

Try $u^*=1$ on $t\in[0,x_0]$, 0 from x_0 on: OK if $x_0\leq T$.

Then $\bar{p}$ must be so that $\bar{p} = e^{-\delta x_0}$, except if $x_0=T$; then any $\bar{p}\in[0,e^{-\delta T}]$ is OK.

But nothing else works: if $e^{-\delta t}$ crosses $\bar{p}$ at, say, t^*, then $u\equiv 1$ on $[0,t^*]$ and 0 from then on, so $x^*(T) = x^*(t^*) = x_0 - t^*$ is >0 if $t^*<x_0$, implying $\bar{p}=0$. But $e^{-\delta t^*} = \bar{p} = 0$ does not have any solution for t^*!

Example 2 Let $\delta \in (0, 1/2)$

$$\max_{u(t)\in\mathbb{R}} \int_0^T e^{-\delta t} u(t)\, dt \quad \text{s.t.} \quad \dot{x} = x - u^2 \qquad \begin{array}{l} x(0) = x_0 \\ x(T) = x_1 \end{array}$$

all constants > 0.

Note: will never choose $u < 0$. (Why not?)

$$H(t, x, u, p) = e^{-\delta t} u + p x - p u^2$$

u^* maximizes H: must have $p > 0$ on $(0, T)$ and $u^* = \dfrac{e^{-\delta t}}{2p}$

$\dot{p} = -\dfrac{\partial H}{\partial x} = -p$ so $p(t) = A e^{-t}$ and thus $A > 0$ and $u^* = \dfrac{e^{(1-\delta)t}}{2A}$

→ Solve for x^*:

$$\dot{x}^* = x^* - e^{2(1-\delta)t} \cdot \frac{1}{4A^2}$$

General sol'n: $x^*(t) = \underbrace{\dfrac{1}{8A^2(\delta - \frac{1}{2})}}_{=B} e^{2(1-\delta)t} + C e^t$

→ Fit constants A, C to x_0 and x_1

$$B + C = x_0$$
$$B e^{2(1-\delta)T} + C e^T = x_1$$

$$\begin{pmatrix} 1 & 1 \\ e^{2(1-\delta)T} & e^T \end{pmatrix} \begin{pmatrix} B \\ C \end{pmatrix} = \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}$$

$$\frac{1}{8A^2} = \frac{x_0 e^T - x_1}{\underbrace{e^T - e^{2(1-\delta)T}}_{}} \left(\delta - \frac{1}{2}\right), \quad C = \frac{x_1 - x_0 e^{2(1-\delta)T}}{e^T - e^{2(1-\delta)T}}$$

OK if thus > 0. Otherwise: no possible sol'n.

Example 3:

$$\max \int_0^T (-x^2 - \dot{x}^2)\, dt \quad , \quad \dot{x} \text{ either } 0 \text{ or } -1$$

$\dot{x} = u$

$x(t_0) = x_0 > 0$ given

$x(T)$ free.

$$H(t,x,u,p) = -x^2 - u^2 + pu$$

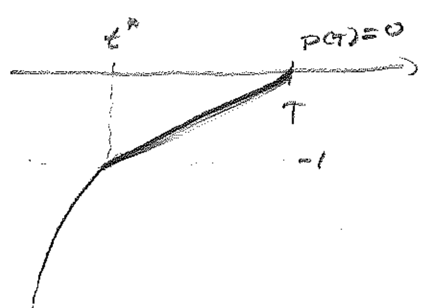
Can choose $u=0$ or $u=-1$, maximizing:

$u=0$ yields $-x^2$ — 0 is optimal if $p > -1$,

$u=-1$ yields $-x^2 - 1 - p$ — -1 is optimal if $p \le -1$.

$$\dot{p} = -\frac{\partial H}{\partial x} = 2x^* = \begin{cases} \text{constant on intervals where } p > -1 \\ \text{decreasing where } p < -1. \end{cases}$$

So $\dot{p}$ concave in t, and



Either $p(t) = \gamma \cdot (t - T)$ everywhere, or from some t^* up.

(In the end, we choose $u^* = 0$!)

- We can have $u^* \equiv 0$ if:

  $x^* \equiv x_0$, $\dot{p} = 2x_0$, $p = 2x_0 \cdot (t - T)$

  $p \ge -1$ always, i.e.: $\underline{2x_0 T \le 1}$.

- Otherwise, choose $u^* = -1$ until $2x^*(t^*)(T - t^*) \le 1$:

  On $[t^*, T]$, $p > -1$ and $u^* = 0$ and $x^* \equiv x^*(t^*) =: \bar{x}$

  $\dot{p} = 2\bar{x}$ yields $p = 2\bar{x} \cdot (t - T)$, equals -1 at t^*

  $2\bar{x}(T - t^*) = 1$.

./.

We can solve out the latter case:

On $[0, t^*]$, $x^* = x_0 - t$

Choose t^* so that

$$2(x_0 - t^*)(T - t^*) = 1$$

There is only one such $t^* \leq \min\{x_0, T\}$.

Note: the maximum principle allows for discrete choices too!

Before suff. cond's: For optimal control, we have

- no "extreme value theorem" existence in this course. ("Filippov - Cesari", if you are interested: FMEA thm 10.4.1 ...)

- no considerations of strict vs non-strict optimum.

Sufficient conditions

Suppose an admissible pair (x^*, u^*) satisfies the conditions from the maximum principle.

When can we know that it solves the max. problem?

Suff. cond's I (Mangasarian): Assume U convex.

The max. principle produces a $p = p(t)$. Plug this into the Hamiltonian to get

$H(t, x, u, p(t))$, a function of (t, x, u).

If this is concave in (x, u), each $t \in (t_0, t_1)$, then (x^*, u^*) is an optimal pair

Suff cond's II (Arrow).

Consider the function $\hat{H}(t, x, p) = \max_{u \in U} H(t, x, u, p)$

(the "maximized Hamiltonian")

Let $p = p(t)$ from the max. principle. Plug into $\hat{H}$ to get $\hat{H}(t, x, p(t))$, a function of (t, x)

If this is concave in x, each $t \in (t_0, t_1)$, then (x^*, u^*) is an optimal pair

Notes $\hat{H}(t, x, p) = H(t, x, \overbrace{\hat{u}(t, x, p)}^{\text{maximizer over } u \in U}, p)$

Arrow is "more powerful": applies to some cases where Mangasarian does not. But may be harder to check.

The examples:

1 $\max_{u \in [0,1]} \int_0^T e^{-\delta t} u \, dt, \quad \dot{x} = -u, \quad u \in [0,1]$

$$H(t,x,u,p) = e^{-\delta t} u - pu$$

Concave wrt (x,u). By Mangasarian, the max. principle is sufficient.

2 $\max_{u \in \mathbb{R}} \int_0^T e^{-\delta t} u \, dt, \quad \dot{x} = x - u^2$

$$H(t,x,u,p) = e^{-\delta t} u + px - pu^2.$$

As long as $p > 0$, Mangasarian $\Rightarrow$ max princ. is sufficient.

→ Had U been closed and bounded, so we could know that a maximizing u^* exists, then it would not depend on x, and $\hat{H}$ would be affine in x and Arrow would apply!

3 $\max_{u \in \{0,-1\}} \int_0^T (-x^2 - u^2) \, dt, \quad \dot{x} = u \in \{0,-1\}$

U not a convex set, so Mangasarian cannot help.

But $H(t,x,u,p) = -x^2 + \underbrace{pu - u^2}_{\text{max of this has no } x}$

So $\hat{H}(t,x,p) = -x^2 + [\text{something with only } p]$

is concave wrt x.

By Arrow, the (x^*, u^*) we found is optimal.